LIMIT THEOREMS FOR NON-COMMUTATIVE OPERATIONS. I.

By Richard Bellman

1. **Introduction.** In this paper a start is made in the construction of a general theory involving the limiting behavior of systems subjected to non-commutative effects.

The classical central limit theorem states that under certain assumptions concerning the distribution function of the independent random variables z_i, the sum $S_N = z_1 + z_2 + \cdots + z_N$ is, when properly normalized, asymptotically Gaussian. This is the mathematical counterpart of the problem of determining the limiting behavior of a physical system subjected to a number of random additive effects.

We propose in a series of papers, of which this is the first, to investigate a number of corresponding problems in which the random effects are not additive, and in particular, non-commutative.

As a simplest example of a problem of this type, let us consider a physical system S to be specified at any time t by a vector $x = (x_1, x_2, \cdots, x_n)$ and to be subject to random sequence of transformations each of which effects a linear transformation upon the state variables, $x \rightarrow Z_k x$, where Z_k is an $n \times n$ matrix. We are interested in the possible limiting behaviors of the system. If the Z_k are all close to the identity matrix, $Z_k = I + \epsilon_k$, the product $\prod_{k=1}^{N} Z_k$ is very nearly $I + \sum_{k=1}^{N} \epsilon_k$ and the problem is again within the commutative domain.

We shall begin with a discussion of this problem for the case where the Z_k are positive matrices. A particular case of this problem connected with the theory of learning processes ([1], [2]) may be treated in a much simpler fashion due to the fact that all the A_k are Markoff matrices.

In subsequent papers we shall discuss the details of this case, various generalizations of the concepts we introduce here, such as the generalized Kronecker power and logarithm of a function, generalizations of the "fundamental identity" of Wald in sequential analysis, generalizations of classical iteration problems, and other related topics.

We have here restricted ourselves to the case of 2×2 matrices in order to reduce the algebraic and notational details which are even in this simplest case occasionally onerous. To further simplify the extraneous details we shall assume that each Z_k is a random matrix possessing the simple distribution

$$\Pr[Z_k = A] = \tfrac{1}{2}, \qquad \Pr[Z_k = B] = \tfrac{1}{2}, \tag{1}$$

Received May 18, 1953.