EUCLIDEAN DOMAINS WITH UNIFORMLY ABELIAN LOCAL FUNDAMENTAL GROUPS, II

By O. G. Harrold, Jr.

It is well known that a closed, topological k-cell in the n-dimensional spherical space S^n may or may not have a complementary space with a trivial fundamental group [2]. It is the purpose of this note to place conditions on the complement of a k-cell C such that $\pi_1(S^n - C) = 0$ and inquire if T is a subset of C, will the complement of T in S^n have a trivial group? If C is an n-cell the answer evidently may be negative by means of the examples parenthetically referred to above. If dimension $C \leq n - 2$, it is almost trivial that if T is closed, a positive answer results, while if dim $C = n - 1$, it is sufficient that T be a *compact absolute retract* in C [3].

This note is to be regarded as a continuation of a paper by the same title [5]. A knowledge of the construction in Theorem IV, $i = n - 1$, is essential for an understanding of the main proof below.

7.1. We recall that $\mathcal{C}(V, U) = 0$ means that if U is a neighborhood of p relative to $A = S^n - C$ then V exists as a neighborhood of p such that all closed paths in V that are homotopic (in U) to a commutator of paths in U are null-homotopic, $\simeq 0$, in U. If for each $p \in S^n$ and each U rel A there exists a V rel A such that $\mathcal{C}(V, U) = 0$, $A = S^n - C$ is said to have *uniformly abelian local fundamental groups.*

Theorem VII. *Let C be a closed, topological k-cell, $k \neq n$, in the n-sphere, S^n, such that $S^n - C$ has uniformly abelian local fundamental groups. If T is any r-cell, $T \subset C$, $r = 1, 2, \cdots, k$, then $\pi_1(S^n - T) = 0$.*

7.2. The proof of the Theorem VII for $k < n - 1$ is a consequence of Theorem IV and the following remark.

If C is a closed subset of S^n such that dim $C < n - 1$ *and* $\pi_1(S^n - C) = 0$, *then for any closed subset $T \subset C$,* $\pi_1(S^n - T) = 0$.

The truth of the remark follows immediately from the fact that if dim $C < n - 1$, $S^n - C$ is connected, locally connected and in fact ULC°. Thus an arbitrary map $f(S^1) = K \subset S^n - T$ may be "rectilinearly" deformed in $S^n - T$ so as to lie in $S^n - C$.

7.3. Before proceeding with the case $k = n - 1$ several preliminary lemmas are needed.

Lemma. *Let C be a closed, topological $n - 1$ cell with boundary $C^{\cdot}$. If T is an absolute retract, $T \subset C$, every component of $C - T$ contains a point of $C^{\cdot}$.*

If t is a homeomorphism of C onto a standard $n - 1$ cell in an $n - 1$ sphere S^{n-1}, $t(T)$ fails to separate S^{n-1} by the fact that T is an absolute retract. Thus

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