DERIVATIVES AND FINITE DIFFERENCES

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The object of the present paper is the proof of some simple formulae concerning derivatives and finite differences of functions of a real variable and of functions of a discrete (integral) variable. As far as they are concerned with functions of continuous variables they may be considered as generalizations of the mean value theorem and of Taylor's formula. The formulae concerning functions of discrete variables are analogues of those on functions of continuous variables.

Both types of formulae can be applied in various ways. The most striking application seems to be a considerably simplified proof of a theorem of Boas and Pólya (§6).

As an application of the "discrete" formulae I prove a theorem of Tauberian character (§8).

1. **Notations and preliminary lemmas.** If $F(x)$ is a function of a real variable and h is a real number, we write

$$\Delta_h^0 F(x) = F(x), \qquad \Delta_h^r F(x) = \sum_{\nu=0}^{r} (-1)^{r-\nu}\binom{r}{\nu} F(x+\nu h) \qquad (r = 1, 2, \cdots)$$

(r-th difference of $F(x)$ with the increment h of the variable x). The same notation will also be used for functions $F(n)$ of a discrete variable n taking integral values only. In this case of course h is also an integer.

If S_n ($n = 1, 2, \cdots$) is a sequence of real numbers (function of the discrete variable n) we write $S_n^{(0)} = S_n$ and

$$S_n^{(r)} = \sum_{\nu=1}^{n} S_\nu^{(r-1)} \qquad (r = 1, 2, \cdots). \tag{1}$$

If the sequence S_n is also defined for $n = 0, -1, -2, \cdots, -(r-1)$, we write

$$S_n^{(-r)} = \Delta_1^r S_{n-r} = \sum_{\nu=0}^{r} (-1)^{r-\nu}\binom{r}{\nu} S_{n-r+\nu} \qquad (r = 1, 2, \cdots).$$

The formula (1) then remains valid in all cases in which the symbols $S_n^{(r)}$ and $S_n^{(r-1)}$ are defined.

We denote by $P_n(r)$ and $Q_n(r)$ the coefficients in the expansions $((e^x - 1)/x)^r = \sum_{n=0}^{\infty} P_n(r)x^n$, $(\log(1 + x)/x)^r = \sum_{n=0}^{\infty} Q_n(r)x^n$. They are polynomials in r, and $P_n(r)$ and $(-1)^n Q_n(r)$ have positive coefficients. (They are related to the

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