CONVERGENCE FACTOR AND REGULARITY THEOREMS FOR CONVERGENT INTEGRALS

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1. **Introduction.** The purpose of this study is the development of some of the relations and theorems for infinite integrals analogous to those obtained for infinite series by Moore [6]. In place of convergent series we consider functions $f(t)$, Lebesgue integrable over $(0, x)$ for every $0 \leq x < \infty$, and for which the limit $\lim_{x\to\infty} \int_0^x f(t)\, dt$ exists. We shall write $\psi(x) = \int_0^x f(t)\, dt$, $\psi = \lim_{x\to\infty} \psi(x)$, and suppose that $| \psi(x) | \leq A < \infty$, $x \geq 0$.

We shall investigate the properties possessed by "convergence factors" $\phi(t, \alpha)$, defined for $t \geq 0$ and α in some convenient set E having a limit point α_0 not of the set. In the work that follows, sets E_1, E_2, E_x, *etc.*, are understood to be subsets of E, each containing all the points of E in a certain neighborhood of α_0. The use of the "join" of two such subsets implies that the join is non-vacuous.

A function $z(t, \alpha)$ defined for t in an interval (a, b) and α in E will be said to converge boundedly as $\alpha \to \alpha_0$ provided that it converges to some limit as $\alpha \to \alpha_0$, and there is a neighborhood of α_0 such that $z(t, \alpha)$ is bounded for all t in (a, b) and all α in this neighborhood.

The factors $\phi(t, \alpha)$ must ensure successively the existence of the following:

$$\sigma(x, f, \alpha) = \int_0^x \phi(t, \alpha) f(t)\, dt \qquad (x \geq 0), \tag{1.1}$$

$$\sigma(f, \alpha) = \int_0^\infty \phi(t, \alpha) f(t)\, dt, \tag{1.2}$$

$$\sigma(f) = \lim_{\alpha\to\alpha_0} \sigma(f, \alpha). \tag{1.3}$$

Finally, we shall demand that ϕ be "regular"; that is, that $\sigma(f) = \psi$.

We consider also the problem of convergence factors $\theta(t, \alpha)$, defined like the $\phi(t, \alpha)$ above, such that

$$\lim_{\alpha\to\alpha_0} \int_0^\infty \theta(t, \alpha) g(t)\, dt = \lim_{t\to\infty} g(t), \tag{1.4}$$

where $g(t)$ is any bounded measurable function defined for $t \geq 0$ for which the right side of (1.4) exists. The discussion of the latter problem yields results similar to those obtained by Agnew [1].

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