THE DETECTION OF THE OSCILLATION OF SOLUTIONS OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION

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Let

$$d[r(x)y']/dx + p(x)y = 0 \tag{1}$$

be a differential equation in which $r(x)$ and $p(x)$ are continuous and $r(x) > 0$ on the interval $0 < x < \infty$. The points $x = 0$ and $x = \infty$ will then, in general, be singular points of (1).

In this paper we study conditions sufficient to insure the existence of an infinity of oscillations in the neighborhood of $x = 0$ and of $x = \infty$. (The reader's attention is directed to the interesting and closely related papers of Wintner and of Hille referred to in the bibliography. The latter paper contains an account of the history of this problem.) Simple conditions of considerable generality are established in §§1 and 2. Necessary conditions which are quite "close" to these are displayed in §3. In §4 a logarithmic scale of tests, each more sensitive than the preceding, is developed. The final section (§5) of the paper is devoted to a brief application to the calculus of variations.

1. **Sufficient conditions.** We introduce the symbols

$$r_0 = \lim_{x \to 0} \int_x^1 r^{-1}(x)\, dx, \qquad p_0 = \lim_{x \to 0} \int_x^1 p(x)\, dx,$$

$$r_\infty = \lim_{x \to \infty} \int_1^x r^{-1}(x)\, dx, \qquad p_\infty = \lim_{x \to \infty} \int_1^x p(x)\, dx.$$

The principal theorem is then the following.

Theorem 1.1. *If $p(x)$ is positive near $x = 0+$ and if $r_0 = +\infty$ and $p_0 = +\infty$, every solution of* (1) *vanishes infinitely often on the interval* $(0, 1)$. *If $p(x)$ is positive near $x = +\infty$ and if $r_\infty = +\infty$ and $p_\infty = +\infty$, every solution of* (1) *vanishes infinitely often on the interval* $(1, \infty)$.

It is clear that the second statement of the theorem will follow from the first if we employ the transformation $x = 1/t$. (Because of this observation we shall state theorems henceforth for only one of the neighborhoods $x = 0+$ or $\infty -$. The reader will readily supply their duals.) Suppose the first statement of the theorem is false. From (1) we have for e positive and sufficiently small $\int_x^e (ry')'y^{-1}\, dx + \int_x^e p\, dx = 0$ where $0 < x \leq e$, and hence

$$r(x)\frac{y'(x)}{y(x)} = r(e)\frac{y'(e)}{y(e)} + \int_x^e r(x)\frac{y'^2(x)}{y^2(x)}\, dx + \int_x^e p(x)\, dx. \tag{1.1}$$

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