THE STRONG LAW OF LARGE NUMBERS

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1. **Introduction.** Among the limit theorems in probability theory which may be regarded as direct descendants of Bernoulli's theorem are the laws of large numbers. These are statements concerning sequences of random variables. In this paper the term *random variable* refers to a real-valued variable $\mathbf{x}$ with which is associated a probability measure

$$P(A) = \Pr \{\mathbf{x} \varepsilon A\}$$

(the probability that $\mathbf{x}$ belongs to A) defined for all Borel sets A in R_1, the space of real numbers, such that $P(A) = 0$ if A is the empty set, and $P(R_1) = 1$. Also associated with a random variable is its distribution function

$$V(t) = \Pr \{\mathbf{x} < t\} = P((-\infty, t)), \tag{1}$$

a non-decreasing function, continuous on the left, with $V(-\infty) = 0$, $V(\infty) = 1$, and

$$P(A) = \Pr \{\mathbf{x} \varepsilon A\} = \int_A dV(t), \tag{2}$$

for each Borel set A in R_1.

If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are random variables, the values of the combined variable $(\mathbf{x}_1, \cdots, \mathbf{x}_k)$ are points in the k-dimensional product space $R_k = R_1 \times R_1 \times \cdots \times R_1$. A probability measure is postulated for all Borel sets in R_k. If $\{\mathbf{x}_n\}$ is a sequence of random variables, the values of the combined variable $\xi = (\mathbf{x}_1, \mathbf{x}_2, \cdots)$ are points in the product space $R_\infty = R_1 \times R_1 \times \cdots$. One may then define a probability measure on the family of all cylinder sets in R_∞, $A = A_k \times R_1 \times R_1 \times \cdots$, where A_k is a Borel set in R_k, by assigning to $P(A) = \Pr \{\xi \varepsilon A\}$ as a value the probability measure in R_k of A_k. As is well known, this probability measure may then be uniquely extended over the minimal Borel extension $\mathfrak{F}$ of the family of all cylinder sets in R_∞.

Only sequences of independent random variables are considered in this paper. A sequence $\{\mathbf{x}_n\}$ is a sequence of independent random variables if for each positive integer n and for all Borel sets $A_1, A_2, \cdots, A_n$ in R_1,

$$\begin{aligned} &\Pr \{\mathbf{x}_1 \varepsilon A_1, \mathbf{x}_2 \varepsilon A_2, \cdots, \mathbf{x}_n \varepsilon A_n\} \\ &\qquad = \Pr \{\mathbf{x}_1 \varepsilon A_1\} \Pr \{\mathbf{x}_2 \varepsilon A_2\} \cdots \Pr \{\mathbf{x}_n \varepsilon A_n\}; \end{aligned} \tag{3}$$

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