A NEW CRITERION FOR THE EXTENSION OF RECTANGLE FUNCTIONS

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Introduction. 0.1. Let $\mathbf{R}$ be any fixed, oriented closed rectangle in the xy-plane, and let $\mathfrak{R}$ be the class of all oriented closed, non-degenerate rectangles contained in $\mathbf{R}$. Denote by $[\phi, \mathfrak{R}]$ a real valued, finite, non-negative function $\phi(R)$ defined for every rectangle R in $\mathfrak{R}$. Reichelderfer and Ringenberg [3] have established the following necessary and sufficient condition that the function $[\phi, \mathfrak{R}]$ admit a non-negative, completely additive extension to an additive class of sets (for definitions of these terms, see [5; 7 and 8]) including all Borel sets: (C) if $R_1, R_2, \cdots$ is a finite or denumerable sequence of mutually exclusive rectangles in $\mathfrak{R}$, and if $R_1^*, R_2^*, \cdots$ is any finite or denumerable sequence of rectangles in $\mathfrak{R}$ such that $\sum_n R_n \subset \sum_m R_m^*$, then $\sum_n \phi(R_n) \leq \sum_m \phi(R_m^*)$.

0.2. Although this result is very useful, it is often difficult to verify an inequality involving two possibly infinite sums. The purpose of this note is to replace condition (C) by equivalent conditions which are easier to apply These conditions are given in

Theorem 1. *A non-negative function $[\phi, \mathfrak{R}]$ of oriented closed rectangles admits a completely additive extension to an additive class of sets including all Borel sets if and only if it satisfies the following three conditions*:

(i) *if R_1 and R_2 are two mutually exclusive rectangles in $\mathfrak{R}$ and R is any rectangle in $\mathfrak{R}$ such that $R_1 + R_2 \subset R$, then $\phi(R_1) + \phi(R_2) \leq \phi(R)$;*

(ii) *if R is any rectangle in $\mathfrak{R}$ and R_1 and R_2 are any two rectangles in $\mathfrak{R}$ such that $R \subset R_1 + R_2$, then $\phi(R) \leq \phi(R_1) + \phi(R_2)$;*

(iii) *if R is any rectangle in $\mathfrak{R}$ and ϵ is a positive number, let R_ϵ denote the common part of $\mathbf{R}$ and the rectangle containing R with sides parallel to the corresponding sides of R at distance ϵ. Then $\lim \phi(R_\epsilon) = \phi(R)$ as ϵ tends to zero through positive values.*

0.3. If the oriented closed rectangles of §0.1 are replaced by oriented open rectangles, the resulting statements remain true; that is, (C) is a necessary and sufficient condition that a non-negative function $[\phi, \mathfrak{R}]$ of oriented open rectangles admit a completely additive extension to an additive class of sets including all Borel sets. Corresponding to Theorem 1, we have

Theorem 2. *A non-negative function $[\phi, \mathfrak{R}]$ of oriented open rectangles admits a completely additive extension to an additive class of sets including all Borel sets if and only if it satisfies the following conditions*:

(i) *same as* §0.2 (i);

(ii) *same as* §0.2 (ii);

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