AN EXAMPLE IN CONFORMAL MAPPING

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1. It is a well-known result in the theory of conformal mapping that if a region R in the w-plane bounded by a closed, rectifiable, Jordan curve C is mapped conformally on a circle $|z| < 1$ by a function $z = f(w)$, analytic and univalent in R, then $f(w)$ will be continuous in the closure of R, possess absolutely continuous boundary values on C as a function of the arc length of C, and will map in a one to one manner sets of points of Lebesgue measure zero on C into sets of Lebesgue measure zero on the circumference $|z| = 1$ and sets of positive Lebesgue measure on C into sets of positive Lebesgue measure on $|z| = 1$. (See [7] or [4; 156–159]. Throughout this paper, measure will always refer to Lebesgue measure.) If the condition that C be rectifiable is dropped, $f(w)$ will be continuous in the closure of R and will map C in a one to one manner on $|z| = 1$. (See [1].) The above metric property, however, can no longer be asserted for $f(w)$ on C. The purpose of this paper is to construct the following example which will be stated in the form of an existence theorem.

Theorem. *There exists a closed Jordan curve C in the w-plane and a point set E on C with the following property: If the interior region r determined by C is mapped conformally by means of a function $z = f(w)$ on $|z| < 1$, the set E goes over into a set of measure zero on $|z| = 1$; if the exterior region R determined by C is mapped conformally by means of a function $z = F(w)$ on $|z| > 1$, the set E goes over into a set of positive measure on $|z| = 1$.*

(R. Nevanlinna [6; 107] states without proof that an example of a set S could be given which belongs to the boundaries of two regions R_1 and R_2, each bounded by a finite number of Jordan arcs, and which is mapped into a set of measure zero in the conformal map of R_1 on the unit circle, while it is mapped into a set of positive measure in the map of R_2 on the unit circle. In our example the two regions are the interior and exterior of the same Jordan curve.)

2. In the sequel, we shall make use of two lemmas. In order to state the first lemma conveniently, a preliminary definition must be introduced.

Definition. A boundary point P of a plane region r will be called *finitely accessible* if there exists a rectifiable, Jordan arc connecting some interior point of r with P and, with the exception of P, lying wholly interior to r.

Lemma 1. *If the circle $|z| < 1$ is mapped conformally on a region r of the w-plane by means of a function $w = f(z)$, analytic and univalent in $|z| < 1$, then all points of $|z| = 1$, with the possible exception of a set of measure zero, will correspond in this map to finitely accessible boundary points of r.*

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