SQUARE ROOT ESTIMATES OF ARITHMETICAL SUM FUNCTIONS

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1. For the remainder term of the sum function of the number of representations of n as a sum of two squares, and for similar functions involving more general non-principal characters, Gauss [3; 362–369], [4; 269–291] found the estimate $O(n^{\frac{1}{2}})$. He did not publish his proofs (cf. Dedekind's commentaries in [4; 292–299]), which were reconstructed, and extended to the hyperbolic case of the divisor problem, by Dirichlet [1; 351–356], [2; 49–66, 97–104]. The method has supplied the estimate $O(n^{\frac{1}{2}})$ in all the problems in question. And nothing better than $O(n^{\frac{1}{2}})$ was available, though better estimates have been claimed, until the beginning of this century.

Subsequently, the application of analytical methods to the class of problems in question improved the exponent $\frac{1}{2}$ of the O-terms to $\frac{1}{3}$. Still later, $\frac{1}{3}$ proved to be replaceable by some $\theta < \frac{1}{3}$ (van der Corput). The optimum, which (except for logarithmic factors) would be $\theta = \frac{1}{4}$, has not been reached in any of the classical problems. And there is a class of lattice problems in which the best exponent proves to be $\theta = \frac{1}{3}$ (Jarnik). For references, cf. [5; 81].

The object of the following considerations is to decide whether the possibility of improving the Gauss-Dirichlet exponent, $\theta = \frac{1}{2}$, is due to an *explicit* nature of their problems or to a *general* arithmetical fact, depending only on that peculiarity of the sequence $1, 2, \cdots, n, \cdots$ which makes possible the application of the underlying sieve process (as to the latter, cf. Dedekind's comments, referred to above). The answer will be that the first of these two possibilities takes place. This means that the classical exponent, $\theta = \frac{1}{2}$, no matter how trivial, cannot in general be improved.

The problem can be reduced to the determination of the order of magnitude of certain "Lebesgue constants". In other words, the result will follow from the fact that certain absolute constants are just of the order of $n^{\frac{1}{2}}$. Incidentally, these constants have a simple arithmetical significance (cf. the deduction of (10) below).

2. The details of the question can best be specified by first considering Gauss' problem of the lattice points in a circle. His result is that, if $f(n)$ denotes the number of the representations of n as a sum of two squares,

$$\sum_{m=1}^{n} f(m) = \pi n + O(n^{\frac{1}{2}}). \tag{1}$$

Geometrically, (1) is almost evident, πn being the area of the circle of radius $n^{\frac{1}{2}}$. Arithmetically, the appearance of the factor π in (1) is due to

$$\pi = 4 \sum_{k=1}^{\infty} (-1)^{k-1}/k, \tag{2}$$

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