# A GENERALIZATION OF BOREL'S AND F. BERNSTEIN'S THEOREMS ON CONTINUED FRACTIONS

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1. **Introduction.** In a recent paper [2] the simple continued fraction was generalized by the following consideration. Let $F$ be the class of real functions $f(t)$, defined for $t \geq 1$ and satisfying the following conditions:

$$f(1) = 1; \tag{1.1}$$

$$f(t_1) > f(t_2) > 0 \qquad (1 \leq t_1 < t_2); \tag{1.2}$$

$$\lim_{t \to \infty} f(t) = 0; \tag{1.3}$$

$$| f(t_2) - f(t_1) | \leq | t_2 - t_1 | \qquad (1 \leq t_1 < t_2); \tag{1.4}$$

there is a constant $\lambda$ such that $0 < \lambda < 1$ and

$$| f(t_2) - f(t_1) | \leq \lambda^2 | t_2 - t_1 | \qquad (1 + f(2) \leq t_1 < t_2). \tag{1.5}$$

If $f \varepsilon F$ then there is a one-to-one correspondence between the numbers $x$, $0 < x < 1$, and all finite and infinite sequences of positive integers. (In the case of a finite sequence the last integer should be greater than unity.) This correspondence is given in the case of a finite sequence $a_1, a_2, \cdots, a_n$ by

$$x = f(a_1 + f(a_2 + \cdots + f(a_n) \cdots)) \tag{1.61}$$

and in the case of an infinite sequence $a_1, a_2, \cdots$ by

$$x = \lim_{n \to \infty} f(a_1 + f(a_2 + \cdots + f(a_n) \cdots)). \tag{1.62}$$

In the future we shall simply write $f(a_1 + f(a_2 + \cdots + f(a_n)$ and $f(a_1 + f(a_2 + \cdots$ for the right members of (1.61) and (1.62), respectively.

If $f(t) = t^{-1}$ the above statements give the well-known results on the expansion of real numbers into simple continued fractions.

In this paper we shall deal with certain measure theoretical problems in the sense of Lebesgue. Therefore, we may disregard the case of a finite sequence $a_1, a_2, \cdots, a_n$ since the corresponding $x$ form a denumerable set, hence a set of measure zero. Moreover, we shall use throughout this paper the notation $a_n(x)$ in order to emphasize the dependence of the $a_n$ on $x$ for a fixed $f \varepsilon F$. Obviously, the $a_n(x)$ are defined for almost all $x$, $0 < x < 1$.

Borel [3] and F. Bernstein [1] have shown in the case $f(t) = t^{-1}$ (simple continued fractions) that (a) the set of all $x$, $0 < x < 1$, for which $a_n(x) \leq k_n$ for all $n$, where the $k_n$ are positive integers, is of measure zero if and only if $\sum 1/k_n$

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