TOPOLOGICAL AND METRICAL POINTS OF VIEW IN THE THEORY OF SETS AND FUNCTIONS OF REAL VARIABLES

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The theory of sets and functions of real variables finds its most important applications to the natural sciences in connection with those functions which are solutions of ordinary differential equations, partial differential equations, or integral equations. These solutions, while regular in the region R where they satisfy the equations in question, generally cease to be regular on the boundary of R. This boundary may result from the nature of the functions themselves, or it may coincide with the locus along which the determining conditions of the solution are supposed given. Moreover, these latter conditions may be incompatible with the regularity of the solution on the boundary of R. Whatever may be the case, the region of validity of the solution is limited by a boundary along which the resolving function ceases to satisfy the given equation and along which it generally loses its character of analyticity or even of continuity. The function considered is often completely characterized by its behavior in the neighborhood of the boundary. That behavior is determined at each point by numbers presenting themselves as limits of expressions which are continuous away from the boundary, but undefined on the boundary itself. These limits, being variable with the point considered on the boundary, are a priori functions of real variables of a fairly general nature, not necessarily analytic, and usually discontinuous. Their study is therefore dependent upon the general theory of functions of real variables. But a fundamental remark is that these characteristic functions present themselves as limits, greatest limits, lowest limits, or merely accumulation values of certain families of functions which depend on parameters, and vary continuously with each of these parameters as long as the parameters remain positively far from the respective limits to which they are tending.

The most interesting questions to be set will often consist of seeking the properties of size (that is, of linear, quadratic, or higher order measure) of the sets E lying on the boundary F, where the numbers characterized above satisfy diverse conditions set a priori.

In this kind of problem, presenting a universal interest in all domains where mathematical analysis finds any application, I would indicate the general methods of reasoning which may be usefully applied. The nature of these methods changes according to the topologic or metric character of the properties to be studied. I shall emphasize the necessity for determining carefully this character in order to avoid seeking uselessly by ineffective methods the answers to questions which must be approached from another point of view.

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