THE EXTENSION OF LINEAR FUNCTIONALS

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1. **Introduction.** If E is a Banach space, its conjugate space E^*, the space of all linear functionals defined on E, is also a Banach space. If $\mathfrak{M}$ is a linear manifold in E, and φ is a linear functional on $\mathfrak{M}$, regarded as a normed linear space in itself, a well known theorem[1] asserts that it is possible to extend φ to all of E without increasing its norm. The extension thus obtained will not, in general, be unique. One of the considerations of this paper is the establishment of criteria for the uniqueness of such extensions in all cases—i.e., for *all* subspaces $\mathfrak{M}$ of the given space. A sufficient condition is found to be that the unit sphere in E^* be strictly convex; this condition is also necessary in case E is reflexive (for details see §4).

Another question which arises is that of whether a rule of extension may be established which will be linear. That is, is it possible to define a *linear* operation A on $\mathfrak{M}^*$ to E^* such that the linear functional $f = A\varphi$ will be an extension of φ, for each φ in $\mathfrak{M}^*$? This will imply, as we show in §3, that $\mathfrak{M}^*$ is isomorphic[2] with a linear subspace in E^*. This question is for the most part distinct from the question of uniqueness of extension and is discussed in §3 with the aid of the notion of a projection of a space on a subspace. §5 is devoted to a few remarks on a situation in which the uniqueness of extension and the existence of the operation A, described above, are bound together. Finally, in §6 an example is given to show the irredundancy of a part of the hypothesis of Theorem 3.

2. **Notation.** Throughout the paper the following conventions in notation will be observed. E denotes a Banach space, E^* its conjugate space, and E^{**} the space conjugate to E^*. $\mathfrak{M}$ denotes a closed linear manifold in E, $\mathfrak{M}^*$ the space conjugate to $\mathfrak{M}$ considered as a space by itself. We use letters $x, y, \cdots$ for elements of E; $f, g, \cdots$ for elements of E^*; $F, G, \cdots$ for elements of E^{**}. Elements of $\mathfrak{M}^*$ are denoted by φ, and elements of $\mathfrak{M}^{**}$ by Φ.

For our purposes it is frequently convenient to introduce the following notation for the values of various linear functionals. We write

$$f(x) = (x, f),$$
$$\varphi(x) = [x, \varphi],$$
$$F(f) = \{f, F\},$$
$$\Phi(\varphi) = \langle \varphi, \langle \Phi ,$$

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[1] S. Banach, *Opérations Linéaires*, p. 55, Théorème 2. We shall refer to this as the Hahn-Banach theorem.

[2] For the definition of this term see the beginning of §3.