THE JACOBI CONDITION AND THE INDEX THEOREM IN THE CALCULUS OF VARIATIONS

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In 1916 Bliss[1] published a paper on the Jacobi condition in which he presented the method now familiar to every student of the calculus of variations. For a parametric problem $\int f(y_1, \cdots, y_n, y'_1, \cdots, y'_n)dt = \text{min.}$ conjugate points on an extremal $y = y(t)$ were defined as zeros of a non-identically vanishing "normal" solution $\eta(t)$ of the Jacobi equations; that is, a solution such that $y'_i(t)\eta_i(t) \equiv 0$.

Hestenes[2] recently commented that, since there are only $2n - 2$ linearly independent normal solutions of the Jacobi equations, Bliss' treatment is quite different from that usually given (i.e., since 1916) for non-parametric problems. He then gave a discussion of conjugate points by adjoining the equation $y'_i\eta'_i =$ constant (or $y'_i\eta''_i + y''_i\eta'_i = 0$) to the Jacobi equations. There are $2n$ linearly independent solutions of the resulting system of equations.

More recently, Birkhoff and Hestenes[3] used the device of adjoining any one of several second-order equations to the Jacobi equations in studying natural isoperimetric conditions, and Morse[4] used one of these equations

$$f_{y^i}\eta^i + f_{y^{i'}}\eta^{i'} = \text{const.}$$

(assuming $f > 0$) in establishing the index theorem.

As compared with earlier treatments of the Jacobi condition, the advantages of Bliss' method are too well known to require mention. But even in comparison with the more recent treatments just mentioned it has the advantage that the property of being a "normal" solution is evidently invariant under change of parameter along the extremal, so that all representations of the curve are equally usable. This is not true of the "special" solutions of Hestenes,[5] and the treatment of Morse rests on a special choice of parameter.

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[1] G. A. Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form*, Trans. Am. Math. Soc., vol. 17(1916), pp. 195–206.

[2] M. R. Hestenes, *A note on the Jacobi condition* $\cdots$, Bull. Am. Math. Soc., vol. 40(1934), pp. 297–302.

[3] G. D. Birkhoff and M. R. Hestenes, *Natural isoperimetric conditions in the calculus of variations*, this Journal, vol. 1(1935), p. 264.

[4] M. Morse, *The index theorem in the calculus of variations*, this Journal, vol. 4(1938), pp. 231–246.

[5] However, it should be remarked that Hestenes suggests several other equally useful forms of adjoined equation, and one of these ($y'_i\eta'_i/y'_jy'_j =$ constant) gives solutions with the invariance property under discussion.