NIL-RINGS WITH MINIMAL CONDITION FOR ADMISSIBLE LEFT IDEALS

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The main theorem of this article is stated in §1 and proved in §2. Possibly the corollaries of this theorem are of more interest than the theorem itself. Let $\mathfrak{O}$ be any ring with minimal conditions for left ideals. From our main theorem it follows that (1) the radical of $\mathfrak{O}$ is nilpotent; (2) the ring $\mathfrak{O}$ is semi-primary (or semi-simple); (3) any subring of $\mathfrak{O}$ containing only nilpotent elements is itself nilpotent. This third corollary is a conjecture of Köthe, which Levitzki[1] proved by assuming both the minimal and maximal condition for right ideals of $\mathfrak{O}$.

1. **Definitions and assumptions.** Let $\mathfrak{N}$ be a nil-ring—i.e., a ring in which every element is nilpotent—and let Ω denote a set of operators for $\mathfrak{N}$, each element of Ω being a left-hand operator for $\mathfrak{N}$. We shall assume that (1) $\mathfrak{N}$ is not the null-ring and that (2) the set Ω contains all the elements of $\mathfrak{N}$. Thus Ω will contain as right-hand operators the elements of $\mathfrak{N}$ (and possibly elements not belonging to $\mathfrak{N}$). We assume the following postulates:

(α_0) $\xi(u + v) = \xi u + \xi v$ for all $\xi \in \Omega$ and u, v in $\mathfrak{N}$;

(α_1) $(\xi\eta)u = \xi(\eta u)$, provided that $\xi\eta$ exists in Ω;

(α_2) $(\xi + \eta)u = \xi u + \eta u$, if $\xi + \eta$ is defined in Ω.

For those elements of Ω which are right-hand operators for $\mathfrak{N}$ we assume the analogues of (α_0)-(α_2) above; e.g., (α_1') asserts that $u(\xi\eta) = (u\xi)\eta$, provided that the product $\xi\eta$ exists in Ω and is a right-hand operator.

If an element θ of Ω is *not* a right-hand operator for $\mathfrak{N}$, we shall need the additional postulate:

(α_3) $\theta(uv) = u(\theta v)$.

At this point we mention three useful relations which are consequences of (2), (α_1), and (α_1') above:

(β) $\xi(uv) = (\xi u)v$;

(γ) $(v\xi)u = v(\xi u)$;

(δ) $(vu)\xi = v(u\xi)$.

We derive (β) from (α_1), and (γ) and (δ) from (α_1'), by regarding the element u of $\mathfrak{N}$ as an operator (see (2) above). Obviously (β) holds for all ξ in Ω, while (γ) and (δ) are valid only when ξ is a right-hand operator. In connection with (α_3) we point out that if ξ is a right-hand operator we do not deny (α_3)—we merely do not assume it.

Received February 28, 1938. I am indebted to members of the Mathematics Seminar of Tulane University for useful suggestions regarding the phraseology of the proofs.

[1] Math. Ann., vol. 105(1931), pp. 620–627.