CRITICAL CURVATURES IN RIEMANNIAN SPACES

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1. **Introduction.** A well known theorem in differential geometry concerns the normal curvatures of curves through a point on a 2-dimensional surface in 3-space. It states that either the curvature is constant, independent of the direction of the curve, or else there is one direction giving a maximum to the curvature, and another (perpendicular) direction giving a minimum. We generalize this result to the case of an n-surface in a Riemannian $(n + 1)$-space. In place of merely a maximum and a minimum, there is in general a non-degenerate critical point of each type or index[1] $0, 1, \cdots, n - 1$. A similar result is obtained for an arbitrary subspace of a Riemannian space, the theorem being stated in terms of projections on any direction orthogonal to the subspace. A final theorem, with a similar statement regarding critical values, concerns the Ricci mean curvature in a Riemannian space.

2. **The principal directions[2] for a real quadratic form.** Our results regarding critical values will be based on the following theorem.

THEOREM 2.1. *Given the real quadratic form*[3]

$$z = a_{ij}x_i x_j, \tag{2.1}$$

on the locus

$$x_i x_i = 1 \tag{2.2}$$

z has at most, and in general exactly, n distinct critical values. When the number is n, the critical values are taken on at n pairs of diametrically opposite points of (2.2), *determining n mutually perpendicular lines through the origin in the number space of the x's. If the pairs are ordered according to the algebraic values of z, at either point of the i-th pair z has a non-degenerate critical point of index i − 1.*

Proof. We begin by making an orthogonal transformation with fixed origin in (x)-space so that the given form becomes (using the same letters $x_1, \cdots, x_n$)

$$z = b_1 x_1^2 + \cdots + b_n x_n^2 \doteq b_i x_i^2 \tag{2.3}$$

with the b's real.[4] Now if we consider the function

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[1] Marston Morse, *The Calculus of Variations in the Large*, Amer. Math. Soc. Colloquium Publications, vol. 18, p. 143.

[2] Cf. L. P. Eisenhart, *Riemannian Geometry*, Princeton, 1926, p. 110. We shall refer to this volume as Eisenhart.

[3] Repetition of an index indicates summation from 1 to n.

[4] Cf. M. Bôcher, *Introduction to Higher Algebra*, p. 170, Theorem 1 and p. 171, Theorem 2; or L. E. Dickson, *Modern Algebraic Theories*, p. 74, Theorem 10.