NOTE ON THE SIMULTANEOUS ORTHOGONALITY OF HARMONIC POLYNOMIALS ON SEVERAL CURVES

By J. L. Walsh and G. M. Merriman

1. In the plane of the complex variable $z = x + iy$, the polynomials $1, z, z^2, \cdots$ are mutually orthogonal, not merely on the circumference $|z| = 1$, but also on every circumference $|z| = R$, in the sense that

$$\int_{|z|=R} z^k \bar{z}^l \, |dz| = 0 \qquad k \neq l.$$

The general problem of the existence of sets of polynomials in z which are simultaneously orthogonal, with respect to suitable norm functions, on each of several curves in the z-plane has been studied only recently. Let us say that the set $p_k(z)$ of polynomials in z is *canonical* on a rectifiable Jordan curve C with respect to the norm function $n(z)$ provided the set $p_k(z)$ is found by orthogonalization on C of the set $1, z, z^2, \cdots$ with respect to the positive continuous norm function $n(z)$, and provided the coefficient of z^k in $p_k(z)$ is chosen positive. Walsh established[1] the orthogonality with respect to a suitable norm function of certain Tchebycheff polynomials on *all* ellipses of a given confocal family. Szegö[2] and Walsh[3] showed independently and by widely different methods the fact that if the same set of polynomials $p_k(z)$ is canonical on two distinct curves C and C', then either C' is a curve C_R or C is a curve C'_R;[4] Szegö requires analyticity of C and C'. [Let C be an arbitrary Jordan curve in the z-plane, and let the function $z = \psi(w)$ map the exterior of C onto the exterior of the unit circle $|w| = 1$ in the w-plane so that the points at infinity in the two planes correspond to each other. We denote generically by C_R the image (Kreisbild) in the z-plane of the circle $|w| = R > 1$ under this transformation.] Moreover, Szegö[5] exhibited all sets of polynomials in z, each set canonical simultaneously on *all* C_R of a given family, $1 < R < \infty$.[6] The general problem of the existence of sets of polynomials canonical simultaneously on only two curves

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1 Bull. Am. Math. Soc., vol. 40 (1934), pp. 84–88. Also *Interpolation and Approximation*, New York, 1935, p. 134, Theorem 12.

2 Trans. Am. Math. Soc., vol. 37 (1935), pp. 196–206.

3 *Interpolation and Approximation*, p. 134, Theorem 11.

4 The analogous result for harmonic polynomials follows directly by the methods of Walsh (loc. cit. and Trans. Am. Math. Soc., vol. 33 (1931), pp. 370–388, especially p. 385).

5 Loc. cit.

6 These sets are enumerated in §2, below.