ASYMPTOTIC LINES THROUGH A PLANAR POINT OF A SURFACE AND LINES OF CURVATURE THROUGH AN UMBILIC

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1. **Introduction.** If a regular point of a surface is not an umbilic, there pass through it two asymptotic lines (which may be imaginary) and two real lines of curvature.[1] If a point is an umbilic, this conclusion does not follow, for a circular umbilic is a singular point for the differential equation of the lines of curvature and a planar umbilic is a singular point for the differential equations of both families of curves. In a previous paper,[2] the author has studied the relations between two finite sets of directions at a planar point: the "true asymptotic directions", which are the possible tangent directions of the asymptotic lines through the point, and the "true principal directions", which are the possible tangent directions of the lines of curvature. In the present paper, we shall consider the asymptotic lines which are tangent to a given true asymptotic direction at a planar point and the lines of curvature which are tangent to a given true principal direction at a planar or circular point.

No previous results for the asymptotic lines appear to be known. The lines of curvature have been studied in the general case by Delloue;[3] the present paper amplifies his conclusions. In special cases, the lines of curvature have been treated by several writers, among whom may be mentioned Cayley,[4] Darboux,[5] Picard[6] and Wahlgren.[7]

The results for the asymptotic lines are not parallel to those for the lines of curvature. To an arbitrary true asymptotic direction at a planar point, there is tangent a unique asymptotic line in the most general case, two asymptotic lines in the next most general case. To an arbitrary true principal direction at a planar or circular point, there is tangent in general either a single line of curvature or an infinite number of lines of curvature, depending upon certain definite conditions.

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[1] An umbilic is a regular point of a surface at which $e/E = f/F = g/G = 1/\rho$, where E, F, G and e, f, g are respectively the coefficients of the first and second fundamental forms of the surface. In the present paper an umbilic will be called a *circular point* if $1/\rho \neq 0$ and a *planar point* if $1/\rho = 0$.

[2] Downs, *Asymptotic and principal directions at a planar point of a surface*, this Journal, vol. 1 (1935), pp. 316–327.

[3] Comptes Rendus, vol. 187 (1928), p. 702.

[4] *On differential equations and umbilici*, Collected Math. Papers, vol. 5, p. 708.

[5] *Leçons sur la Théorie Générale des Surfaces*, vol. 4, Note VII.

[6] *Traité d'Analyse*, vol. 3, chap. IX, §14.

[7] Arkiv for Mat., Astr. och Fys., vol. 1 (1903), p. 43.