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ERGODIC MEASURES, ALMOST PERIODIC POINTS AND DISCRETE ORBITS

BY

MICHEL TALAGRAND

Let K be a compact space, and ρ be a homeomorphism of K. A set $L \subset K$ is said to be *invariant* if $\rho L \subset L$, and is said to be *minimal* if it is closed, invariant and minimal with respect to these two properties.

A point $\omega \in K$ is said to be *almost periodic* if for each neighborhood V of ω, the set $\{i \in \mathbf{N};\ \rho^i\omega \in V\}$ is relatively dense in $\mathbf{N}$. Denote by A^ρ the set of all almost periodic points. It is known that $\omega \in A$ if and only if the closure of $\{\rho^i(\omega);\ i \geq 0\}$ is minimal.

A point $\omega \in K$ will be said to be *recurrent* if it is not almost periodic and if each neighborhood V of ω, the set $\{i \in \mathbf{N};\ \rho^i(\omega) \in V\}$ is infinite. Denote by R^ρ the set of recurrent points, and denote by D^ρ the complement of $R^\rho \cup A^\rho$, that is the set of points whose orbit is discrete. The sets A^ρ, R^ρ, D^ρ are invariant.

Denote by M^ρ the set of all ρ-invariant Radon probabilities on K. It is a convex w^*-compact set, and an invariant probability μ on K is said to be *extremal* if it is extremal in M^ρ. For $\mu \in M^\rho$ and X any subset of K, we denote by $\mu^*(X)$ and $\mu_*(X)$ the outer and inner measure of X. If μ is extremal and X invariant, then for all X we have $\mu^*(X), \mu_*(X) \in \{0, 1\}$.

Let us denote by τ the map $n \to n+1$ from $\mathbf{N}$ to $\mathbf{N}$, and again by τ the restriction to $\beta\mathbf{N}\backslash\mathbf{N}$ of its canonical extension to the Stone-Čech compactification $\beta\mathbf{N}$ of $\mathbf{N}$.

In [1], very interesting results concerning A^τ, R^τ, D^τ are proved. Our aim is to investigate, from a slightly different point of view, for an extremal $\mu \in M^\rho$, what can be the inner and outer measure of A^ρ, R^ρ, D^ρ. If the support supp μ is minimal, it is contained in A^ρ. So we have to investigate only what happens if this support is not minimal. Let

$$E^\rho = \{\mu \in M^\rho\colon \mu \text{ is extremal, supp } \mu \text{ is not minimal}\}.$$

The following result shows that if $\mu \in E^\rho$, then A^ρ is small for μ.

THEOREM 1. *Let $\mu \in E^\rho$. Then $\mu^*(A^\rho) = 0$.*

Proof. Let F be the support of μ. If F is not minimal, F contains an invariant closed G such that $G \neq F$. Let U be an open set of K such that $U \cap F \neq \phi$, $\bar{U} \cap G = \phi$. For all n let $V_n = \bigcup_{i \leq n} \rho^i(U)$. Since $V_n \cap G = \phi$, V_n

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