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PARTIALLY ISOMETRIC APPROXIMATION OF POSITIVE OPERATORS

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1. Introduction

Consider the problems of minimizing the quantity

$$\|A - U\|_p$$

where A is a fixed positive operator and where U varies over the set of (i) all unitaries, (ii) all isometries, and (iii) all partial isometries (subject to the condition that $A - U \in \mathscr{C}_p$ where $\mathscr{C}_p$ denotes the von Neumann-Schatten p class). In the language introduced by Halmos [6], problems (i), (ii) and (iii) concern, respectively, unitary, isometric and partially isometric approximants in $\mathscr{C}_p$ of a positive operator. Problem (i) has been solved by Aiken, Erdos and Goldstein [1]. This paper tackles problems (ii) and (iii).

Aiken, Erdos and Goldstein proved that if the operator A is positive and U varies over all those unitaries such that $A - U \in \mathscr{C}_p$, where $1 \leq p < \infty$, then $\|A - U\|_p$ is minimized when $U = I$ and, providing the underlying Hilbert space is finite-dimensional, maximized when $U = -I$ [1, corollary 3.6]. Further, if A is strictly positive and $1 < p < \infty$ these minimum and maximum points are unique [1, Theorem 3.5]. They also obtained the corresponding inequality for the operator norm [1, Theorem 3.1]: if A is positive then for all unitaries U in $\mathscr{L}(H)$

$$\|A - I\| \leq \|A - U\| \leq \|A + I\|. \tag{1.1}$$

A feature of their work is the use of noncommutative differential calculus. They found an explicit formula [1, Theorem 2.1] for the derivative of the map $X \mapsto \|X\|_p^p$, where $X \in \mathscr{C}_p$ with $1 < p < \infty$ (see Theorem 2.3 below). In searching for a global minimizer of $\|A - U\|_p$ one can thus restrict attention

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