RAMIFICATION INDEX AND MULTIPLICITY

BY

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Introduction

Consider the family consisting of all pairs of valuation rings $R \subset S$ such that S dominates R and the field of quotients of S is a finite algebraic extension of the field of quotients of R. Then for each such pair $R \subset S$, the ramification index $r(S/R)$ is by definition $e(S/R)[\bar{S}:\bar{R}]_i$, where $[\bar{S}:\bar{R}]_i$ is the degree of inseparability of the residue class field extension $\bar{S} \supset \bar{R}$ and $e(S/R)$ is the reduced ramification index which is the index of the value group of R in that of S (see [8, pp. 50–82]). This integer-valued function on this family has the following basic properties:

(1) $r(S/R) = 1$ if and only if S is unramified over R.

(2) $r(T/R) = r(T/S)\cdot r(S/R)$.

(3) $r(S/R) = [S:R]$ (the degree of the field extension of the field of quotients of R) if S is a finitely generated R-module and the residue class field extension $\bar{R} \subset \bar{S}$ is purely inseparable.

Our main purpose in this paper is to show that there exists one and only one integer-valued function $r(S/R)$ having the above properties, where R and S are local, integrally closed noetherian domains instead of valuation rings. Before we can give a more detailed account of the main results, we need some definitions which will hold throughout the rest of the paper.

By a ring we mean a commutative, noetherian ring with a unit element 1 different from zero. If R is a ring, a ring S together with a ring homomorphism $f : R \to S$ such that $f(1) = 1$ will be called an *R-algebra*. An R-algebra S will be called a *local R-algebra* if R and S are local rings and if there is an R-algebra A which is finitely generated as an R-module such that S is isomorphic, as an R-algebra, to $A_{\mathfrak{M}}$ for some maximal ideal $\mathfrak{M}$ in A. Thus if S is a local R-algebra, then the residue class field extension $\bar{R} \subset \bar{S}$ is of finite degree, and $\mathfrak{m}S$ is an ideal of definition of S where $\mathfrak{m}$ is the maximal ideal of R (i.e., $\mathfrak{m}S$ contains some power of the maximal ideal of S). Since $\mathfrak{m}S$ and $\mathfrak{m}$ are ideals of definition in S and R, we can talk about $e_S(\mathfrak{m}S)$, the multiplicity in the sense of Samuel (see [7] or [8, VIII, Section 10]) of $\mathfrak{m}S$ in S, and $e_R(\mathfrak{m})$, the multiplicity of $\mathfrak{m}$ in R. The rational number $e_S(\mathfrak{m}S)/e_R(\mathfrak{m})$ will be called the multiplicity or reduced ramification index of the local R-algebra S and will be denoted by $e(S/R)$. In analogy with the valuation ring situation, we define the ramification index $r(S/R)$ of the local R-algebra S to be $e(S/R)[\bar{S}:\bar{R}]_i$.

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