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A MORSE THEORY FOR EQUIVARIANT YANG-MILLS

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0. Introduction. This paper develops a Morse theory for equivariant Yang-Mills and Yang-Mills-Higgs fields on compact Riemannian 4-manifolds. One immediate consequence is the first proof of the existence of nontrivial Yang-Mills-Higgs fields on S^4. Another is a topological criteria for the existence of non-self-dual Yang-Mills fields: for appropriate Lie group actions, if the space of equivariant connections does not retract to the equivariant moduli space, then there exist Yang-Mills connections that are neither SD nor ASD (Theorem 3.4).

The Yang-Mills equations arise as a variational problem. Let $P \to M$ be a principal G-bundle over a compact Riemannian manifold. Each connection A on P has a curvature 2-form F^A. The Yang-Mills action

$$YM(A) = \int_M |F^A|^2 \, dv \tag{0.1}$$

is a function on the space $\mathscr{A}$ of connections whose critical points are the YM fields. More generally, we can (following the physicists) choose a hermitian vector bundle E associated to P and define, for each connection A on P and each section ϕ of E, the Yang-Mills-Higgs action

$$YMH(A, \phi) = \frac{1}{2} \int_M |F^A|^2 + |d^A\phi|^2 + \frac{1}{2}(|\phi|^2 - \mu)^2 \, dv \tag{0.2}$$

where $\mu > 0$ is a constant, the "mass parameter". The critical points are Yang-Mills-Higgs (YMH) fields. (The last term, the "Higgs potential", is included with the aim of ensuring that the Lagrangian has nontrivial minima; without it, YMH would be minimized by $\phi \equiv 0$.) Both (0.1) and (0.2) are invariant under the gauge group $\mathscr{G}$ and hence descend to functions on the orbit spaces $\mathscr{B} = \mathscr{A}/\mathscr{G}$ and $\mathscr{E} = \mathscr{A} \times_{\mathscr{G}} \Gamma(E)$.

One can view this situation from the perspective of Morse theory. When M is 4-dimensional, the Yang-Mills function on $\mathscr{B}$ is minimized along the moduli space $\mathscr{M}$ of self-dual/anti-self-dual connections. Other, nonminimal critical points have recently been discovered ([SSU], [SS], [Pk3]). A Morse theory for YM could provide a conceptually simple method for obtaining such nonminimal critical YM fields. A Morse theory for the YMH action would be even more useful because no nontrivial YMH fields are known. Unfortunately, the usual formulations of Morse

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