A NOTE ON THE COMPOSITION OF ARITHMETIC FUNCTIONS

By L. Carlitz

1. Let $F(n)$, $G(n)$ be arithmetic functions of the positive integral variable n. If

$$\Psi(n) = \lim_{N=\infty} \sum_{k=1}^{N} F(k)G(k+n) \qquad (n = 1, 2, 3, \cdots) \tag{1}$$

exists, Rearick [4] calls Ψ the cross-correlation of F and G.

Put

$$F(n) = \sum_{d \mid n} df(n), \qquad G(n) = \sum_{d \mid n} dg(n),$$

thus defining the arithmetic functions $f(n)$, $g(n)$.

For given η, $0 \leq \eta < 1$, let $C(\eta)$ denote the class of all arithmetic functions F such that the associated functions f satisfy the conditions

$$nf(n) = O(n^{\epsilon}) \qquad (\text{for all} \quad \epsilon > 0), \tag{2}$$

$$\sum_{n \leq x} n|f(n)| = O(x^{\eta}). \tag{3}$$

Then by a theorem of Mirsky [2, Theorem 2], if F and G belong to the class $C(\eta)$, it follows that $\Psi(n)$ exists for all $n \geq 1$; moreover

$$\psi(n) = \sum_{\substack{a,b=1 \\ (a,b)=n}}^{\infty} f(a)g(b), \tag{4}$$

where

$$\Psi(n) = \sum_{d \mid n} d\psi(d). \tag{5}$$

The present writer [1] had earlier proved the existence of $\Psi(n)$ when $F(n)$ and $G(n)$ satisfied somewhat stronger conditions than (2) and (3). Put

$$F(n) = \sum_{d \mid n} df(d), \qquad f_1(a) = \sum_{k=1}^{\infty} f(ak);$$

as noted by Rearick, the absolute convergence of $\sum_{k=1}^{\infty} f(k)$ is a consequence of (3). It was proved in [1] that

$$F(n) = \sum_{k=1}^{\infty} f_1(k)c_k(n), \tag{6}$$

where

$$c_k(n) = \sum_{\rho(k)} \rho^n,$$

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