THE VANISHING OF RAMANUJAN'S FUNCTION $\tau(n)$

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The numerical function $\tau(n)$ of Ramanujan defined by

$$\prod_{\nu=1}^{\infty} (1 - x^{\nu})^{24} = \sum_{n=1}^{\infty} \tau(n) x^{n-1} \tag{1}$$

has been the subject of numerous investigations since Ramanujan in 1916 first discovered its remarkable properties (see [4; Chapters 9 and 10]). This function is generated by the 24-th and 8-th powers respectively of the lacunary power series of Euler and Jacobi

$$\prod_{\nu=1}^{\infty} (1 - x^{\nu}) = \sum_{r=-\infty}^{\infty} (-1)^r x^{(3r^2+r)/2}, \tag{2}$$

$$\prod_{\nu=1}^{\infty} (1 - x^{\nu})^3 = \sum_{s=0}^{\infty} (-1)^s (2s + 1) x^{s(s+1)/2},$$

and it is natural to ask whether $\tau(n) = 0$ for any $n > 0$. The recent discovery of D. F. Ferguson (communicated by a letter of August 3, 1946) that the 53rd coefficient of the 15-th power of (2) is zero adds to the interest in the question of the possible vanishing of $\tau(n)$. Tables of $\tau(n)$ given in [6], which extend to $n = 300$ show no case of $\tau(n) = 0$. In this paper we show that $\tau(n) \neq 0$ for $n < 3316799$. Whether τ (3316799) $= 0$ or not we cannot say. The methods of this paper would seem to be incapable of establishing that $\tau(n)$ is never zero. That such a simple question about a well-known function is apparently difficult to answer is due to the fact that no practicable formula for $\tau(p)$ (p a prime) has ever been discovered.

1. In our discussion we use the following formulas and congruence properties. The numbers in square brackets indicate items in the bibliography where the corresponding results are proved.

$$\tau(m)\tau(n) = \tau(mn) \qquad (m, n \text{ coprime}) \qquad [7], [4] \tag{3}$$

$$\tau(p^{\alpha}) = \tau(p)\tau(p^{\alpha-1}) - p^{11}\tau(p^{\alpha-2}) \qquad (p \text{ a prime}) \qquad [7], [4] \tag{4}$$

$$\tau(p^{\alpha}) = p^{11/2} \csc \theta_p \sin (\alpha + 1)\theta_p \qquad [7], [4] \tag{5}$$

where $2 \cos \theta_p = \tau(p)p^{-11/2}$. If we use $\sigma_k(n)$ to denote the sum of the k-th powers of the divisors of n, we have

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