THE ASYMPTOTIC DISTRIBUTION OF RUNS OF CONSECUTIVE ELEMENTS

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In a permutation of $1, 2, \cdots, n$ let r denote the number of instances in which i is next to $i + 1$, i.e., in which either of the successions $(i, i + 1)$ or $(i + 1, i)$ occurs. Thus for the permutation 234651, $r = 3$. In [3] Wolfowitz[1] has proposed the use of r for significance tests in the non-parametric case, and in [4] he has shown that asymptotically r has the Poisson distribution with mean value 2. It is to be noted that $W(R)$, the number of runs as defined by Wolfowitz, is equal to $n - r$.

In this note we shall derive more explicit results concerning the asymptotic distribution of r. In a random permutation (all permutations being regarded as equally probable) let the probability of exactly r successions as above be $P(n, r)$, and let $M(n, k)$ denote the k-th factorial moment of the distribution, that is

$$M(n, k) = \Sigma_r r(r - 1) \cdots (r - k + 1)P(n, r).$$

We shall show that

$$(1) \qquad M(n, k) = 2^k \left[1 - \frac{k+1}{2k}\binom{k}{1}\frac{k}{n} + \frac{k+2}{2^2 k}\binom{k}{2}\frac{k(k-1)}{n(n-1)} - \cdots\right]$$

$$(2) \qquad P(n, r) = \frac{2^r e^{-2}}{r!}\left[1 - \frac{r^2 - 3r}{2n} + \frac{r^4 - 8r^3 + 9r^2 + 22r - 16}{8n(n-1)}\right] + 0(n^{-3}).$$

Since 2^k is the k-th factorial moment of the Poisson distribution with mean 2, either of these results serves to verify the asymptotic Poisson character of the distribution of r.

It would be possible to obtain some kind of explicit formula for the general term of (2), but there seems to be no reasonably simple form.

Proof of (1). Let A_i denote the event "$i + 1$ comes right after i" and B_i the event "i comes right after $i + 1$" $(i = 1, \cdots, n - 1)$. The joint probability of k of these $2n - 2$ events is either 0, if they are incompatible, or $(n - k)!/n!$ if they are compatible—for in the latter case we in effect assign positions for k of the elements and are then free to permute the $n - k$ others. Let $f(n, k)$ denote the number of ways of selecting k compatible events. Then it is known that ([1], eq. (40))

$$(3) \qquad M(n, k) = k!f(n, k)(n - k)!/n! = f(n, k)\Big/\binom{n}{k}.$$

[1] I am indebted to Dr. Wolfowitz for calling my attention to this problem, and to its identity with what I called the "n-kings problem" in [2].