BOUNDS ON A DISTRIBUTION FUNCTION WHEN ITS FIRST n MOMENTS ARE GIVEN[1]

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Introduction. Let $F(X)$ be a nondecreasing function defined on the real line with $F(-\infty) = 0$ and

$$\int_{-\infty}^{\infty} t^k \, dF(t) = M_k \qquad k = 0, \cdots, 2n.$$

Then the problem of Tchebycheff is to find upper and lower bounds for $F(X)$. If X is a random variable with the cumulative distribution function $F(X)$, this is just the problem of determining the (sharp) upper and lower bounds for Pr. $(X \leqq d)$. This problem has been solved by Markoff [2] and Stieltjes [5] and their results are given in Section 1.

It is often of interest, however, to determine upper and lower bounds for Pr. $(|X| \leqq d)$. This is the problem of determining upper and lower bounds on the cumulative distribution function $F^*(X^*) = F(X^*) - F(-X^*)$ of the nonnegative random variable $X^* = |X|$, and leads to the Stieltjes moment problem: To determine upper and lower bounds on the nondecreasing function F^* given

$$\int_{-\infty}^{\infty} t^k \, dF^*(t) = M_k \qquad k = 0, \cdots, n;$$

and $F^*(0-) = 0$. The numbers M_k are now the absolute moments of X, that is $M_k = E(|X|^k)$. It should be noted that the set of moments.

$$M_{\alpha k} = E(|X|^{\alpha k}) \qquad k = 0, \cdots, n,$$

serves just as well as $M_0, \cdots, M_n$, since they are the first n algebraic moments of the nonnegative random variable $Y = |X|^\alpha$ with the cumulative distribution function $G(Y) = F(Y^{1/\alpha})$.

In the second and third section we give a solution to this problem which corresponds to the classical Tchebycheff inequalities for the Hamburger moment problem, and apply these general results in the next section to obtain the Cantelli inequalities. I would like to point out that Theorems 1 and 2 can be derived from very general results of Krein [9]. However, the self-contained approach used here seems to me desirable in view of the complexity and inaccessibility of Krein's results. In the last section we solve the problem of determining sharp upper and lower bounds for a distribution given the first two (absolute) moments about the mode.

1. The Tchebycheff inequalities. A point t is said to belong to the spectrum of the random variable X or of the corresponding distribution function $F(X)$

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