DISTRIBUTIONS OF THE MEMBERS OF AN ORDERED SAMPLE

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1. Introduction. Let the members of a random sample from a distribution $F(x)$ with probability density $F'(x) = f(x)$ be in order of magnitude $x_1, \cdots, x_m, \cdots, x_n, \cdots, x_N$, with $x_i \leqq x_{i+1}$, $i = 1, \cdots, N - 1$, and $m < n$. We shall compute the moments of the distribution of x_m and of the joint distribution of x_m and x_n.

The results are derived under the assumption that $F^{-1}(x)$, the inverse of $F(x)$, is a polynomial. Then we discuss the applicability of the results to any distribution for which $F^{-1}(x)$ is differentiable at $m/(N + 1)$ and $n/(N + 1)$. In this general case no restriction on $F(x)$ is imposed other than the differentiability; in particular, the interval on which $0 < F(x) < 1$ can be finite, semi-finite, or infinite.

2. Present status of the problem. This problem is handled through analyses of several specific distributions in reference [1] listed at the end of this paper. It is suggested that any one of the Pearson type frequency curves can be adequately approximated by one of the density functions handled in that paper. Although a general method is employed, there is no general development or general results; each distribution requires special, extensive computations. In contrast to these earlier results, the present paper contains a general development with results that are easily specialized to particular distributions.

Following [1] there have been discussions of asymptotic distributions. It is known that if m and N increase with m/N approaching a limit different from zero and one, under quite general conditions the distribution of x_m is asymptotically normal; see [2] or [3]. Also it was pointed out in [4] that with some restrictions on the distribution function the limiting distribution of x_m as N increases, but m is fixed, has the probability density

$$m^m \exp\,[my - \exp\,(-y)]/(m - 1)!$$

where y is a normalization of x_m; see [5]. However, it is suggested in [6] that in the case of the normal distribution if $m = 1$, one should have a sample of size 10^{12}, and Mr. Kendall concludes in [5], p. 221, that "For practical purposes, therefore, there is still no adequate general approximate form for the distribution of mth values." However, a contribution to the asymptotic case of this problem is made in [6]. In contrast to these asymptotic results, the present paper is concerned with the exact sampling distributions for any sample size. In the case of large samples, known approximations concerning moments are equivalent to the leading terms of some of the expansions of this paper.

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