NOTES

DISTRIBUTION OF LINEAR CONTRASTS OF ORDER STATISTICS[1]

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Introduction. Many theoretical and practical problems of statistical nature have lead investigators to study methods capable of pooling the information contained in the ordered (or ranked) sample values with some properties of the assumed distribution of the parent population. Since, in analysis of variance situations, contrasts between functions of observations are of utmost importance, linear contrasts of order statistics will be considered here under the assumption that the underlying distribution is normal.

Null distribution of linear contrasts of order statistics. Let $x_0, x_1, \cdots, x_n$ denote $n + 1$ independent normal random variables with unknown means $\mu_1, \mu_2, \cdots, \mu_n$ respectively, and with a common variance $\sigma^2 = 1$ (say). Let $x_{(0)} > x_{(1)} > \cdots > x_{(n)}$ be the ordered values. Consider the following linear contrast

$$z = x_{(0)} - c_1 x_{(1)} - c_2 x_{(2)} - \cdots - c_n x_{(n)}, \qquad \sum_{i=1}^{n} c_i = 1;$$
$$0 \leqq c_i \leqq 1, \quad i = 1, \cdots, n.$$

Using, as a starting point, the joint density of $x_{(0)}, x_{(1)}, \cdots, x_{(n)}$ as given by Wilks [7], and with the help of appropriate transformations, the null distribution of z can be obtained. It takes the form of a rather messy expression containing a n-fold iterated integral. An interesting particular case: the density of the difference between the two largest ordered values can be obtained from the general form. St-Pierre and Zinger [6] have tabulated the null density of $u = x_{(0)} - x_{(1)}$ using a slightly different method.

It is of interest to consider the above contrast in the case of three random variables. The density of $z = x_{(0)} - cx_{(1)} - (1 - c)x_{(2)}$, under the hypothesis H_0: $\mu_0 = \mu_1 = \mu_2 = 0$ (say), takes the form

$$g(z) = 3[\pi(c^2 - c + 1)]^{-1/2} \exp [-z^2/4(c^2 - c + 1)] \cdot \int_{(2c-1)z/[6(c^2-c+1)]^{1/2}}^{(c+1)z/(1-c)[6(c^2-c+1)]^{1/2}} (2\pi)^{-1/2} \exp(-t^2/2)\, dt. \tag{1}$$

With the help of [3], [4], and [5], $g(z)$ can be tabulated. Values of $g(z)$ are given in Table I for several values of the parameter c.

From the general form (1), several densities can be derived as particular cases. For instance, the value $c = 0$ leads to the density of the range as given by McKay

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