# NOTES

## A PROOF OF WALD'S THEOREM ON CUMULATIVE SUMS

By N. L. Johnson

*University College London*

**1. Introduction.** In the theory of sequential analysis developed by Wald [1], there occurs a theorem, one form of which can be expressed as follows:

THEOREM 1. *If*

(i) $z_1, z_2, z_3 \cdots$ *are independent random variables with common expected value* $\mathcal{E}(z) = \mu$,

(ii) $\mathcal{E}(|z_i|) \leqq A < \infty$ *for all* $i$, *and some finite* $A$,

(iii) $n$ *is a random variable taking values* $1, 2, 3, \cdots$ *with probabilities* $P_1, P_2, P_3 \cdots$ *respectively, and*

(iv) *the event* $\{n \geqq i\}$ *depends only on* $z_1, z_2, \cdots z_{i-1}$,

*then, setting* $Z_n = \sum_{i=1}^n z_i$,

$$\mathcal{E}(Z_n) = \mu\mathcal{E}(n).$$

This note presents a simple proof of this theorem. It appears to be an abbreviated form of an argument due to Wolfowitz [2].

In Sections 3 and 4 of this note an extension of the method to the evaluation of the variance of $n$ is discussed.

**2. Proof of Theorem 1.** Let $y_i = 1$ if $z_i$ is observed (i.e. if the event $\{n \geqq i\}$ occurs) and $y_i = 0$ if $\{n \geqq i\}$ is not observed, so that

$$\Pr\{y_i = 1\} = \Pr\{n \geqq i\} = \sum_{j=i}^\infty P_j .$$

Then $Z_n = \sum_{i=1}^\infty y_i z_i$ and $\mathcal{E}(Z_n) = \mathcal{E}(\sum_{i=1}^\infty y_i z_i) = \sum_{i=1}^\infty \mathcal{E}(y_i z_i)$ since $\sum_{i=1}^\infty |\mathcal{E}(y_i z_i)| < A\mathcal{E}(n) < \infty$. By reason of (iv),

$$\mathcal{E}(y_i z_i) = \mathcal{E}(y_i)\mathcal{E}(z_i),$$

so

$$\mathcal{E}(Z_n) = \sum_{i=1}^\infty \mathcal{E}(y_i)\mathcal{E}(z_i) = \mu \sum_{i=1}^\infty \mathcal{E}(y_i),$$

$$= \mu \sum_{i=1}^\infty (P_i + P_{i+1} + \cdots) = \mu \sum_{i=1}^\infty iP_i = \mu\mathcal{E}(n).$$

**3. An analogous second moment theorem.**

THEOREM 2. *If we assume, in addition to* (i)-(iv), *that*

(v) $\text{var}(z_i) = \mathcal{E}(z_i^2) - \mu^2 = \sigma^2 < \infty$, *with the same value for all* $i$,

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