THE RANDOM WALK BETWEEN A REFLECTING AND AN ABSORBING BARRIER

By B. Weesakul

The University of Western Australia

1. Introduction. In this paper, the classical problem of random walk restricted between two barriers at 0 and b is discussed. A particle, starting from the initial position u on the x-axis ($0 < u \leqq b$ an integer) at $t = 0$, moves one unit to the left or right of its position at times $t = 1, 2, \cdots$. The probabilities for the moves are respectively q and $p(q + p = 1)$, the moves being independent. We assume that the barrier at 0 is absorbing and the one at b reflecting so that (i) when the particle reaches the barrier at 0, it is absorbed and the process terminates (ii) when at any integral time $\tau(\tau \geqq b - u)$, the particle is at the barrier at b, there is a probability p that it remains there at the next instant $(\tau + 1)$ and a probability q that it moves one unit to the left.

Random walk problems have been extensively studied (see Feller [1]), and their application to the theory of Brownian movement has been discussed by Kac [2] among others. With the assumption that there is one reflecting barrier at 0 and the other at ∞, Kac was able to derive an explicit expression for

$$P(n, m \mid s),$$

the probability that the particle starting from position n is at m after time s has elapsed. Other cases where both barriers are absorbing and where both barriers are reflecting have also been discussed by Feller [1]. We are concerned in this paper with the case where one barrier is absorbing and the other reflecting; we shall derive the expression for the generating function of the probabilities of absorption.

2. Generating function for the probabilities of absorption. Let $g(t \mid u)$ be the probability that the particle reaches the barrier at 0 for the first time (thus being absorbed) at time t starting from the initial position u at $t = 0$. The probability $g(t \mid u)$ satisfies the difference equation:

$$g(t \mid u) = g(t - 1 \mid u - 1)q + g(t - 1 \mid u + 1)p, \qquad (u = 1, 2, \cdots, b - 1; t = 1, 2, \cdots), \tag{1}$$

where $g(0 \mid 0) = 1$ and $g(t \mid u) = 0$ for $t < u$. For $u = b$, we have

$$g(t \mid b) = g(t - 1 \mid b - 1)q + g(t - 1 \mid b)p.$$

Let $P(u)$ be the $1 \times b$ row vector $(0 \cdots 0\, q\, 0\, p\, 0 \cdots 0)$ with q being the $(u - 1)$th component, and let $G(t - 1)$ be the $b \times 1$ column vector of elements $g(t - 1 \mid i)$, $(i = 1, 2, \cdots, b)$. Then equation (1) may be written in the matrix

Received August 19, 1960.