A CHARACTERIZATION OF THE MULTIVARIATE NORMAL DISTRIBUTION[1]

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1. Introduction: independence of linear forms. Let $X_1, \cdots, X_n$ be independent p-dimensional random row vectors, and let there exist non-zero constants $a_1, \cdots, a_n, b_1, \cdots, b_n$, such that $\sum X_i a_i$ is independent of $\sum X_i b_i$. By considering all linear combinations $\theta X_i'$, where $\theta = (\theta_1, \cdots, \theta_p)$, it follows from the well-known univariate result, first proved completely by Skitovič [7], that the X_i are normally distributed. (For a history of the subject, see Lukacs [4, Section 5].) However, when the scalars a_i, b_i are replaced by $p \times p$ matrices A_i, B_i, this reduction to the univariate case no longer holds. The matrix case for $n = 2$ was treated in [2]. In this paper we treat the general multivariate case.

Another peculiarity of the matrix case stems from the distinction between singularity and vanishing of a matrix. In the one-dimensional problem, if one of the coefficients a_i or b_i is zero, the distribution of the corresponding random variable can be completely arbitrary. The same is true in the matrix case if one of the matrices A_i or B_i is zero. However, if a matrix A_i, say, is singular but not zero, then some linear combinations of elements of the corresponding random vector X_i are normally distributed, but the distribution of X_i is partly arbitrary. An example of a possible consequence is the following:

Let X_1, X_2 be independent random row vectors, and let A be a singular matrix of rank r such that $X_1 + X_2$ and $X_1 + X_2 A$ are independent. There exist non-singular matrices M and N such that $A = M\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}N$, where I_r is the identity matrix of order r. Writing

$$X_i M = Y_i = (Y_{i1}, Y_{i2}), \quad \text{and} \quad NM = B = (B_{ij}), i, j = 1, 2,$$

we have that $(Y_{11}, Y_{12}) + (Y_{21}, Y_{22})$ is independent of $(Y_{11}, Y_{12}) + (Y_{21}B_{11}, Y_{21}B_{12})$. Consequently, the hypothesis does not restrain Y_{22} sufficiently to determine its distribution, and in fact, if Y_{22} is independent of Y_{21}, it can have any distribution without affecting the hypothesis.

We now state the principal result and outline its proof. The main details, which have an intrinsic interest, are given in the next section.

Theorem. *Let $X_1, \cdots, X_n$ be n mutually independent p-dimensional random*

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