PAIRWISE COMPARISON AND RANKING IN TOURNAMENTS

By Hans Bühlmann[1] and Peter J. Huber[2]

University of California, Berkeley

1. Introduction. This paper is concerned with the following ranking problem: $n \geqq 3$ items are compared pairwise. The results of all comparisons can be summarized in a *preference matrix* $A = (a_{ij})$ where $a_{ij} = 1, 0$, or $\frac{1}{2}$, respectively, according as item i is preferred to j, item j is preferred to i or no preference is expressed between i and j, respectively. Which is the best method of ranking all items in the "order of their preferences" provided A is known?

In tournaments of chess, which represent a canonical model for the above situation, it is customary to rank in descending order of the *scores* $s_i = \sum_j a_{ij}$. Since, however, other ranking procedures have been proposed, e.g., by Wei-Kendall [3], the problem arises how to characterize the "goodness" of any such procedure. In this paper we give such a characterization in terms of the "underlying probability structure" and then exhibit a class of such structures for which the usual ranking procedure by *scores* s_i is optimal.

In order to keep what follows as intuitive as possible we shall from now on use the terminology referring to chess tournaments, i.e., "player" for "item," "game" for "comparison" and "won," "lost" or "drawn" for the possible results of any comparison.

2. The underlying probability structure and the correct ranking.

2.1. In probabilistic terms a tournament can be described as follows. The a_{ij}'s appearing in the matrix A are considered as random variables which take on the values 0, $\frac{1}{2}$, and 1. For the purposes of this paper we shall, however, exclude draws, i.e., we assume for all $i \neq j$

$$[a_{ij} = 1] \text{ with probability } p_{ij}, \tag{1}$$

$$[a_{ij} = 0] \text{ with probability } q_{ij} = p_{ji}, \tag{2}$$

and

$$p_{ij} + p_{ji} = 1. \tag{3}$$

The case where draws are permitted will be discussed in a forthcoming paper by Huber [2] where our results are extended to more general random variables a_{ij}.

If the results of all games are independent, then the *probability matrix* $P = (p_{ij})$ describes the complete underlying *probability structure* of the *preference matrix* A, whose elements can then be looked upon as follows: (i) a_{ij} for $i < j$

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