COMBINATORIAL RESULTS IN MULTI-DIMENSIONAL FLUCTUATION THEORY

By Charles Hobby[1] and Ronald Pyke[2]

University of Washington

Combinatorial lemmas have been used quite successfully in analyzing sums of random variables. However, except for the work of Baxter [1], all the results of which the authors are aware have been restricted to the case of real numbers and real variables. In the present paper we examine one possible m-dimensional analogue of the combinatorial results contained in [2]. Other generalizations using rectilinear regions have been attempted but have been found not to lead to invariant results. Our main combinatorial result is given in Theorem 1. An application of this result to multidimensional stochastic processes is given in Theorem 2.

Let $c = (c_1, \cdots, c_n)$ be a sequence of n real numbers. Denote by $\mathcal{S}_c$ the set of all $2^n n!$ sequences which can be formed from c by permuting the numbers c_i and by assigning a $+$ or a $-$ sign to each of the c_i. If $x \varepsilon \mathcal{S}_c$, set $s_0(x) = 0$ and $s_i(x) = x_1 + \cdots + x_i$ for $i = 1, 2, \cdots, n$. We say that the sequence x has type (m, k) if exactly m of the partial sums $s_i(x)$ are greater than 0 and exactly k of the partial sums $s_i(x)$ are less than $s_n(x)$. Set $v_n(m, k; c)$ equal to the number of sequences $x \varepsilon \mathcal{S}_c$ which have type (m, k). The sequence c is said to possess *property* D if for every x in $\mathcal{S}_c$, $s_i(x) \neq 0$ for $i = 1, 2, \cdots, n$. In an earlier paper, the authors have shown (Theorem 2.1 of [2]) that $v_n(m, k; c)$ is independent of c if c possesses property D. More precisely, if c possesses property D, then

$$v_n(m, k; c) = \binom{2m}{m}\binom{2k}{k} 2^{n-1-2m-2k}(n-1)! \tag{1}$$

for all $0 \leqq m, k \leqq n$ satisfying $m + k < n$. Furthermore, $v_n(n - m, n - k; c) = v_n(m, k; c)$.

It is shown in Corollary 3.2 of [2] that the above result (1) implies that the number of sequences $x \varepsilon \mathcal{S}_c$ which have exactly r partial sums $s_i(x)$ in the interval bounded by $s_0(x)$ and $s_n(x)$ is independent of c. In seeking an m-dimensional analogue of this result, we first observe that an interval may be regarded as a one-dimensional sphere. The event of a partial sum falling between s_0 and s_n in the one-dimensional case might then become, in the m-dimensional case, the event of a partial sum falling within the sphere which has the line segment joining

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