BOUNDS FOR DISTRIBUTIONS WITH MONOTONE HAZARD RATE, II

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1. Introduction. In the preceding paper, which we will refer to as "I", we have derived improvements of Markov's inequality under the condition that F has a monotone hazard rate. These results are based on the assumption that for some monotone function ζ, $\int_{0-}^{\infty} \zeta(x)\, dF(x) = \nu$ is known (typically, $\zeta(x) = x^r$, in which case we denote ν by μ_r). In Section 3 of the present paper, we derive similar inequalities for the case that μ_1 and μ_2 are both known. These results may be regarded as improvements, made possible by the assumption of a monotone hazard rate, of the following inequality given by Chebyshev (1874): If F is a probability distribution such that $F(0-) = 0$ and $\int_0^\infty x^r\, dF(x) = \mu_r$, $r = 1, 2$ and $\mu_1 = 1$, then

$$(1.1)\qquad \begin{aligned} 1 - F(t) &\leqq 1, && 0 \leqq t \leqq 1\\ &\leqq t^{-1}, && 1 < t < \mu_2\\ &\leqq (\mu_2 - 1)/[\mu_2 - 1 + (t-1)^2], && t \geqq \mu_2\,; \end{aligned}$$

$$(1.2)\qquad \begin{aligned} 1 - F(t) &\geqq (1-t)^2/[\mu_2 - 1 + (1-t)^2], && 0 \leqq t \leqq 1\\ &\geqq 0, && t \geqq 1. \end{aligned}$$

Improvements of (1.1) and (1.2) have also been obtained by Royden (1953), who assumed that F is concave on $[0, \infty)$ (see (6.1) and (6.2)).

The method used in this paper differs from those of I and can be utilized to provide alternate proofs of the results given there. However, we have not been able to obtain the results of this paper by the more straightforward methods of I.

In Section 5, we again consider the problem discussed in I of improving Markov's inequality, but in this paper we assume that F has a density f which is a Pólya frequency function of order 2 (PF_2). Again, the methods of I do not seem to be useful, but the result is obtained by a method similar to that used in Section 3.

Throughout this paper we assume unless otherwise stated that distribution functions are left continuous.

2. The method of proof. Let $\mathfrak{F}$ be a family of probability distributions. Call $\mathcal{G} \subset \mathfrak{F}$ extremal for $\mathfrak{F}$ on T if for each $t \in T$ and $F \in \mathfrak{F}$, there exists $G \in \mathcal{G}$ such that $F(t) = G(t)$. If $\mathcal{G}$ is extremal for $\mathfrak{F}$ and $F \in \mathfrak{F}$, then clearly

$$\inf_{\mathcal{G}} G(t) \leqq F(t) \leqq \sup_{\mathcal{G}} G(t).$$

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