A LOCAL LIMIT THEOREM FOR NONLATTICE MULTI-DIMENSIONAL DISTRIBUTION FUNCTIONS[1]

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1. Introduction and statement of results. Local limit theorems for asymptotically stable lattice distribution functions have been obtained by Gnedenko [2], [3] for the one-dimensional case and by Rvačeva [6] for the multi-dimensional case. We shall here obtain analogous results for nonlattice distribution functions.

Let F be a stable distribution function in d-dimensional space R^d which has a density p. Let F_1 be a distribution function in the domain of attraction of F, let F_n denote the n-fold convolution of F_1 with itself, and let B_n and A_n be constants in R and R^d respectively such that

$$(1) \qquad \lim_{n\to\infty} F_n(B_n(x + A_n)) = F(x), \qquad x \,\varepsilon\, R^d.$$

Let f and f_1 denote the characteristic functions of F and F_1 respectively. We say that F_1 is *nonlattice* if

$$(2) \qquad |f_1(\theta)| < 1, \qquad \theta \,\varepsilon\, R^d - (0).$$

We say that F_1 is *strongly nonlattice* if

$$(3) \qquad e^{-c_1 d} = \limsup_{|\theta|\to\infty} |f_1(\theta)| < 1.$$

It is clear that F_1 is nonlattice if it is strongly nonlattice.

Note that lattice distribution functions and nonlattice (as defined here) distribution functions do not exhaust all possibilities unless $d = 1$, since a distribution function can be lattice in some directions, but not in others. The last possibility will not be considered in this paper.

For $x = (x^1, \cdots, x^d) \,\varepsilon\, R^d$ and $h > 0$, let $P(x, h)$ and $P_n(x, h)$ denote the measures assigned by F and F_n respectively to the set

$$\{y = (y^1, \cdots, y^d) | x^k \leqq y^k < x^k + h \quad \text{for} \quad 1 \leqq k \leqq d\}.$$

If $d = 1$, for example, then $P(x, h) = F(x + h) - F(x)$; while if $d = 2$, then

$$P(x, h) = F(x^1 + h, x^2 + h) - F(x^1 + h, x^2) - F(x^1, x^2 + h) + F(x^1, x^2).$$

It follows from (1) that

$$(4) \qquad \lim_{n\to\infty} P_n(B_n(x + A_n), B_n h) = P(x, h), \quad x \,\varepsilon\, R^d \quad \text{and} \quad h > 0.$$

The purpose of this paper is to prove the following

Theorem. *If F_1 is nonlattice, then*[2]

$$(5) \qquad P_n(B_n(x + A_n), B_n h) = P(x, h) + o_n(1)(h^d + B_n^{-d}).$$

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[2] We use in this paper the convention that the behavior of any "o" term is uniform in all variables not listed in the term or previously fixed.