FINE STRUCTURE OF THE ORDERING OF PROBABILITIES OF RANK ORDERS IN THE TWO SAMPLE CASE[1]

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1. Introduction. In constructing admissible two sample rank order tests one needs information on the ordering of probabilities of rank orders. Specifically, if, under some restriction of the class of alternatives, the rejection region of a test contains the rank order z then it should contain all rank orders more probable than z.

This paper contains several theorems on such orderings under various alternatives, especially the location parameter case for symmetric distributions.

2. Notation and assumptions. $X = (X_1, \cdots, X_m)$ and $Y = (Y_1, \cdots, Y_n)$ are samples drawn from absolutely continuous populations with densities $f(\cdot)$ and $g(\cdot)$, respectively. $F(\cdot)$ and $G(\cdot)$ denote the corresponding distributions. $W = (W_1, \cdots, W_{m+n})$ denotes the order statistics of the combined sample, $(X, Y) = (X_1, \cdots, X_m, Y_1, \cdots, Y_n)$, and $Z = (Z_1, \cdots, Z_{m+n})$ is a random vector of zeros and ones whose ith component, Z_i, is 0 if W_i comes from $f(\cdot)$ and 1 if W_i comes from $g(\cdot)$.

Let $z = (z_1, \cdots, z_{m+n})$ be a fixed vector of zeros and ones; we define the *complement of* z, $z^c = (z_1^c, \cdots, z_{m+n}^c)$ and the *transpose of* z, $z^t = (z_1^t, \cdots, z_{m+n}^t)$, to be the vectors whose ith components are $1 - z_i$ and $z_{m+n+1-i}$, respectively. $P(z) = \Pr\{Z = z\}$ denotes the probability of the rank order z.

Since the following restrictions of f and g are assumed in several results below, we list them now along with a shorthand notation.

Restrictions. ST: $f(x) = f(-x)$ and $g(x) = f(x - \theta)$, where θ is a non-negative constant.

U: $f(x) \geqq f(x')$ if $0 \leqq x < x'$ or $x' < x \leqq 0$.

MLR: $g(y)/f(y) \geqq g(x)/f(x)$ if $x \leqq y$.

N: $f(\cdot)$ and $g(\cdot)$ are normal densities with common variance 1 and means 0 and θ, respectively, where $\theta \geqq 0$.

Note. ST stands for *S*ymmetry and *T*ranslation and U implies that $f(\cdot)$ is *U*nimodal. It is assumed, without loss of generality, that the mode of $f(\cdot)$ is the origin. MLR stands for *M*onotone *L*ikelihood *R*atio and N stands for *N*ormality. Of course N is the strongest and implies the other three.

3. Theorems on the ordering of rank order probabilities. The general expression for $P(z)$ is

$$P(z) = m!\,n! \int \cdots \int \textstyle\prod_{i=1}^{m+n} h_{z_i}(t_i)\, dt_i, \tag{3.1}$$

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