A REMARK ON SEQUENTIAL DISCRIMINATION[1]

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1. Introduction. You are observing data gathered sequentially. The data is governed by one of a countable number of hypotheses. When all the data is in, these hypotheses are completely distinguishable. At some stage, depending only on data you have in hand, you want to stop and decide which hypothesis is correct, with the probability of error arbitrarily small. You can do this if and only if, for each hypothesis, there is a test based on a bounded amount of data which distinguishes that hypothesis from the set of all the others, with the probability of error arbitrarily small. The object of this note is to state and prove this theorem. (Since countable additivity does not simplify the problem, it will not be assumed, except as noted.)

Here is a special case of the theorem. A coin is to be tossed independently and repeatedly. The probability of heads is unknown, but is known to lie in a countable set Θ. You can stop, and decide what the parameter is with arbitrarily high probability, if and only if each point of Θ is isolated (that is, there is an interval around it free of other points of Θ).

2. Statement of theorem. Let Ω be a set. For $n = 1, 2, \cdots$ let $\mathcal{A}_n$ be a field of subsets of Ω, with $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$. Let $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, a field. Let Π be a countable set of finitely additive probabilities P on $\mathcal{A}$. A stopping time τ on Ω is a function taking the values $1, 2, \cdots, \infty$, with $\{\tau = n\} \varepsilon \mathcal{A}_n$ for $n = 1, 2, \cdots$. Let $\mathcal{A}_\tau$ be the field of all subsets A of Ω such that $A \cap \{\tau = n\} \varepsilon \mathcal{A}_n$ for $n = 1, 2, \cdots$. Unfortunately, $\mathcal{A}_\tau$ is not in general a subfield of $\mathcal{A}$. Define $P_*\{\tau < \infty\}$ as $\lim_{n\to\infty} P\{\tau \leqq n\}$ and $P_*(A)$ as $\lim_{n\to\infty} P(A \cap \{\tau \leqq n\})$ for $A \varepsilon \mathcal{A}_\tau$.

Consider the following two conditions:

(I) For any $\epsilon > 0$, there is a stopping time $\tau = \tau_\epsilon$, and there are disjoint sets $A_P \varepsilon \mathcal{A}_\tau$ for $P \varepsilon \Pi$, with $P_*\{\tau < \infty\} = 1$ and $P_*(A_P) > 1 - \epsilon$ for all $P \varepsilon \Pi$.

(II) For any $P \varepsilon \Pi$ and $\epsilon > 0$, there is a set $A = A_{P,\epsilon} \varepsilon \mathcal{A}$, such that $P(A) \geqq 1 - \epsilon$ and $Q(A) \leqq \epsilon$ for all $Q \varepsilon \Pi - \{P\}$.

Theorem 1. *Conditions* I *and II are equivalent.*

Condition I was introduced for an example by H. Robbins. I learned it from D. Blackwell. After seeing a draft of this paper, Robbins informed me that he had previously obtained Theorems 1 and 4.

The results of this paper can be viewed as extending some previous work of Hoeffding and Wolfowitz (1958).

3. Proof of Theorem 1. The first step is to prove (I) $\Rightarrow$ (II). Fix $P \varepsilon \Pi$ and $\epsilon > 0$. Find a stopping time τ and disjoint sets $A_Q \varepsilon \mathcal{A}_\tau$ with $Q_*\{\tau < \infty\} = 1$

Received 18 January 1967.

[1] This paper has been supported by the National Science Foundation, Grant GP-2593.