PRESERVATION OF WEAK CONVERGENCE UNDER MAPPINGS

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Suppose we have a weakly convergent sequence of probability measures defined on some space S and that we carry the measures over to another space S' by means of a measurable mapping, or perhaps by means of a whole sequence of mappings. How far is weak convergence preserved?

Throughout what follows, S and S' denote two separable metric spaces. The letter h will always denote a measurable mapping from S into S' (Borel measurability), the letter g will denote a measurable mapping from S' into the reals $\dot{R}$, P will be used for a probability measure on S, and Q will be used for a probability measure on S'. Weak convergence of a sequence of probability measures, notationally indicated by the symbol $\rightarrow_w$, is defined in the usual way requiring convergence of the integrals for every real, bounded and continuous function.

If h is a P-continuity function (i.e. continuous a.e. P) and if $P_n \rightarrow_w P$ then weak convergence is preserved, i.e. $P_n h^{-1} \rightarrow_w Ph^{-1}$. This is almost trivial and one would guess that the P-continuity of h is also necessary for the preservation of weak convergence; indeed, this is so as demonstrated in [4].

A more complicated problem arises if, instead of one h, we have a whole sequence $\{h_n\}$ of mappings and ask whether $P_n h_n^{-1} \rightarrow_w Ph^{-1}$ holds for every sequence $\{P_n\}$ with $P_n \rightarrow_w P$. A powerful sufficient condition has been given by Rubin in an unpublished paper ([5]). Here we shall find necessary and sufficient conditions.

Since it is of no importance that the limit measure in the above formulation of the problem is generated from P via a mapping h, we shall replace it by a measure Q. To be precise, we are given a sequence of mappings $\{h_n\}_{n \geqq 1}$, a probability measure P, and a probability measure Q; and we search after conditions that $P_n h_n^{-1}$ converges weakly to Q whenever P_n converges weakly to P. When this holds, we shall say that *weak convergence is preserved*; from the context it should always be clear which mappings and measures we have in mind. Clearly, weak convergence is preserved iff

$$(1) \qquad \forall_{g \text{ bd.cont.}}, \quad \forall_{P_n \rightarrow_w P} \int g(h_n)\, dP_n \rightarrow \int g\, dQ$$

holds (bd. cont. = "bounded continuous").

For every fixed g we can solve the problem suggested by (1). If f is a function from S into $\dot{R}$ and if δ and ϵ are positive, we denote by $\partial_{\delta,\epsilon}(f)$ or $\partial_{\delta,\epsilon} f$, the *$\delta,\epsilon$-boundary* of f, the set of those points x in S for which the distance between $f(x')$ and $f(x'')$ exceeds ϵ for some pair of points x', x'' in the open sphere with center x and radius δ (see [4], [6]).

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