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UNBIASEDNESS OF SOME TEST CRITERIA FOR THE EQUALITY OF ONE OR TWO COVARIANCE MATRICES[1]

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1. Introduction. The main purpose of this paper is to answer the question stated in Anderson and Das Gupta [2], of whether the modified likelihood ratio test (= modified LR test) for the equality of two covariance matrices is unbiased or not. We shall answer this question affirmatively in Section 3, by generalizing the method in Pitman [6]. The same idea can be applied to prove the unbiasedness of the modified LR test for the equality of a covariance matrix to a given one in Section 2 and also of the LR test for sphericity in Section 4, for the equality of a mean and a covariance matrix to some given ones in Section 5. Some generalizations of these results will be also stated.

The derivation of these test criteria can be found in Anderson [1]. Gleser [4] has proved recently the unbiasedness of the LR test for sphericity by reducing the problem to the unbiasedness of the Bartlett test in case of the equal sample sizes. But our method of proof is more direct and somewhat different from his.

2. Unbiasedness of the modified LR test for $\Sigma = \Sigma_0$. Let $p \times 1$ vectors $X_1, X_2, \cdots, X_N$, $(N > p)$, be a random sample from a multivariate normal distribution with unknown mean vector μ and unknown covariance matrix Σ (nonsingular). From this sample we want to test the hypothesis $H_1 : \Sigma = \Sigma_0$ against the alternatives $K_1 : \Sigma \neq \Sigma_0$, where the mean μ is unspecified and Σ_0 is a given $p \times p$ positive definite (= pd) matrix. The acceptance region of the LR test for this problem is given by, as in Anderson ([1], p. 265),

$$\omega_1' = \{S \mid S \text{ is pd and } |S\Sigma_0^{-1}|^{N/2} \operatorname{etr}(-\tfrac{1}{2}\Sigma_0^{-1}S) \geqq c_\alpha\}, \tag{2.1}$$

where the symbol etr means exp tr, $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$, $\bar{X} = N^{-1}\sum_{\alpha=1}^{N} X_\alpha$ and the constant c_α is determined such that the level of this test is α. In case $p = 1$, this acceptance region ω_1' does not give an unbiased test and further the UMP unbiased test is given by replacing $|S\Sigma_0^{-1}|^{N/2}$ to $|S\Sigma_0^{-1}|^{(N-1)/2}$ in (2.1), which can be seen, for example, in Lehmann ([5], p. 165) by some calculation. After this modification of changing the sample size N to the degrees of freedom $N - 1 = n$, we can prove the unbiasedness in the multivariate case. This is the simplest case in our discussion.

Theorem 2.1. *For testing the hypothesis $H_1 : \Sigma = \Sigma_0$ against the alternatives $K_1 : \Sigma \neq \Sigma_0$ for unknown mean μ, the modified* LR *test having the following ac-*

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