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SOME MULTIVARIATE COMPARISON PROCEDURES BASED ON RANKS

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1. Introduction. There are the p-variate treatment populations π_i with the cdf $F_i(\mathbf{x})$, $i = 1, \cdots, c$ and a p-variate control π_0 with $F_0(\mathbf{x})$ where we assume that $F_0(\mathbf{x}) = F(\mathbf{x})$, $F_i(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\theta}_i)$, $\boldsymbol{\theta}_i' = (\boldsymbol{\theta}_i^{(1)}, \cdots, \boldsymbol{\theta}_i^{(p)})$, $i = 1, \cdots, c$ and $F(\mathbf{x})$ is continuous, but unknown otherwise. Now set $\boldsymbol{\Delta}_i' = (\Delta_i^{(1)}, \cdots, \Delta_i^{(l)})$, $\Delta_i^{(h)} = \mathbf{a}_h'\boldsymbol{\theta}_i$, $i = 1, \cdots, c$, $h = 1, \cdots, l$ for the l given constant vectors $\mathbf{a}_h' = (a_h^{(1)}, \cdots, a_h^{(p)})$ where the matrix $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_l)$ has rank l, $l \leqq p$. Then the criterion for the goodness of the treatment is defined as follows:

(i) the control π_0 is best if $\boldsymbol{\Delta}_i \leqq \mathbf{0}$ for $i = 1, \cdots, c$;

(ii) π_i is better than the control π_0 if $\boldsymbol{\Delta}_i \geqq \mathbf{0}$ and $\boldsymbol{\Delta}_i \neq \mathbf{0}$ hold;

(iii) π_i is not better at the $\mathbf{h}$th component than the control π_0, $\mathbf{h} = (h_1, \cdots, h_t)$, $1 \leqq h_1 < \cdots < h_t \leqq l$, $1 \leqq t \leqq l$, if $\Delta_i^{(h_\alpha)} < 0$ and $\Delta_i^{(h_\beta)} \geqq 0$ hold for $\alpha = 1, \cdots, t$ and $\beta \notin \{1, \cdots, t\}$ where generally $\mathbf{x} \leqq \mathbf{y}$ means that each component of $\mathbf{x}$ is not larger than the corresponding component of $\mathbf{y}$ and $\mathbf{0}$ means the zero vector.

If $l = p$, and $\mathbf{A} = \mathbf{I}$, the identity matrix, then (i)–(iii) reduce to comparing the p components of $\boldsymbol{\theta}_i$ with the p vector zero, $i = 1, \cdots, c$. Then a multivariate comparison problem to separate the better treatments than π_0 from the not better ones may be introduced. Some similar problems have been respectively dealt with by Krishnaiah-Rizvi [2] under the assumption of the normal populations, and by Tamura [4], [5] under the nonparametric circumstances. This paper attempts some generalization for them. A formulation for the above problem and some lemmas are given in Section 2. The procedures based on (a) the randomized normal score statistics, (b) the statistics of Wilcoxon type, and (c) the classical sample means will be respectively proposed in Section 3 and their properties will be also investigated in this section.

2. Some lemmas. Though the following Lemma 1 is elementary, it plays an important part for our formulation.

LEMMA 1. *Let the cdf of the random vector* $\mathbf{X}$ *of p-variates be* $F(\mathbf{x} - \boldsymbol{\theta})$ *with the pdf* $f(\mathbf{x} - \boldsymbol{\theta})$ *and covariance matrix* $\boldsymbol{\Sigma}$ *where* $\boldsymbol{\theta}' = (\theta^{(1)}, \cdots, \theta^{(p)})$. *Then the pdf of the random vector* $\mathbf{Y}' = (Y^{(1)}, \cdots, Y^{(l)})$, $Y^{(h)} = \mathbf{a}_h'\mathbf{X}$ *for* $h = 1, \cdots, l$, *is given by the form* $g(\mathbf{y} - \boldsymbol{\Delta})$ *where* $\boldsymbol{\Delta} = (\Delta^{(1)}, \cdots, \Delta^{(l)})$, $\Delta^{(h)} = \mathbf{a}_h'\boldsymbol{\theta}$ *with the covariance matrix* $\mathbf{A}'\boldsymbol{\Sigma}\mathbf{A}$.

PROOF. Without loss of generality assume that $[A_1] = [a_i^{(h)}]$, $i, h = 1, \cdots, l$, and $|A_1| \neq 0$. Then by the transformation

$$\begin{bmatrix} \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \\ \mathbf{0} & \mathbf{I}_{p-l} \end{bmatrix} \mathbf{X}$$

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