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A NOTE ON THRIFTY STRATEGIES AND MARTINGALES IN A FINITELY ADDITIVE SETTING[1]

By William D. Sudderth

Morehouse College and the University of Minnesota

1. Introduction. Let F be a set. Denote by $P(F)$ the set of all finitely additive probability measures defined on all subsets of F. A strategy σ is a sequency $\sigma_0, \sigma_1, \cdots$ where σ_0 is in $P(F)$ and, for all $n > 0$, σ_n is a map from $F \times \cdots \times F$ (n-factors) to $P(F)$. The strategy σ may be viewed as the distribution of a stochastic process $f_1, f_2, \cdots$. That is, σ_0 is the distribution of f_1 and $\sigma_n(f_1, \cdots, f_n)$ is the conditional distribution of f_{n+1} given $(f_1, \cdots, f_n)$. A theory of integration with respect to a strategy σ was developed by Dubins and Savage in [2]. The notation used here is taken mostly from that source.

Let Q_0 be a constant and, for $n > 0$, let Q_n be a real-valued function defined on $F \times \cdots \times F$ (n-factors). Suppose the Q_n are uniformly bounded and, for every $n > 0$ and every n-tuple $(f_1, \cdots, f_n)$,

$$(1) \quad \int Q_n(f_1, \cdots, f_{n-1}, f_n)\, d\sigma_{n-1}(f_1, \cdots, f_{n-1})(f_n) \leqq Q_{n-1}(f_1, \cdots, f_{n-1}).$$

Then the sequence $Q = \{Q_n\}$ is said to be an expectation decreasing semi-martingale with respect to σ ([2], page 29).

If t is a stop rule and $h = (f_1, f_2, \cdots)$, set

$$(2) \qquad Q_t(h) = Q_{t(h)}(f_1, \cdots, f_{t(h)})$$

and

$$(3) \qquad Q(\sigma, t) = \int Q_t\, d\sigma.$$

(The integral is defined in [2] and in Section 2 below.) If $t \leqq t'$, one can show as in [2], page 43, that $Q(\sigma, t) \geqq Q(\sigma, t')$. Define $Q_\infty = \lim_{t\to\infty} Q(\sigma, t)$.

In this note necessary and sufficient conditions are given for a semi-martingale Q to satisfy $Q_\infty = Q_0$. In a countably additive setting a semi-martingale process Q for which $Q_\infty = Q_0$ is a martingale in the sense that equality holds in (1) almost surely. (Example 3.6.2 of [2] shows that this need not be true in our more general setting.) Conditions corresponding to those of this note are easy to find for countably additive processes and are given in [1], page 311. Here conditions are also given for a semi-martingale to be almost a martingale in the sense that $Q_\infty \geqq Q_0 - \epsilon$. As an application, a characterization is given of thrifty strategies for gambling problems, thus solving a problem left open in [2].

2. A slight extension of the Dubins and Savage Integral. Let σ be a strategy and let g be a bounded, real-valued, finitary function defined on

$$H = F \times F \times \cdots.$$

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