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CONSISTENT ESTIMATES OF THE PARAMETERS OF A LINEAR SYSTEM[1]

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1. Introduction. We will be concerned with the following dynamic linear system which finds application in both economics and engineering, for example Aoki [3] and Griliches [6] have used this model.

$$x_{k+1} = Ax_k + v_k, \qquad k \geqq 0 \tag{1.1}$$

$$y_k = x_k + w_k, \qquad k \geqq 1. \tag{1.2}$$

In (1.1), the state equation, x_k is a p-dimensional column vector which represents the state of some system at time k; A is a $p \times p$ transition matrix; and v_k represents a random disturbance, or noise.

In (1.2), the observation equation, y_k represents an observation made on the system at time k, and w_k represents noise. We will assume that $v_0, v_1, \cdots$ and $w_1, w_2, \cdots$ are independent sequences of zero mean, independent and identically distributed random vectors with covariance matrices V and W respectively and that x_0 is independent of the v_i's and w_j's and has finite covariance matrix. We remark, in passing, that the superficially more general model in which (1.2) is replaced by

$$y_k = Mx_k + v_k, \qquad k \geqq 1,$$

where M is nonsingular, may be reduced to (1.2) by an appropriate change of bases.

When A, V, W, and the distribution of x_0 are known, linear least squares prediction and filtering may be done with the Kalman Filter [10], which provides a method for computing the projections, $x_{t \mid k}$ and $y_{t \mid k}$, of x_t and y_t on the Hilbert subspace spanned by $y_1, \cdots, y_k$. Specifically,

$$\begin{aligned} x_{k \mid k} &= (I - \Delta_k)Ax_{k-1 \mid k-1} + \Delta_k y_k, && k \geqq 1, \\ x_{t \mid k} &= A^{t-k}x_{k \mid k}, && \\ y_{t \mid k} &= x_{t \mid k}, && t > k, \end{aligned} \tag{1.3}$$

where I denotes the $p \times p$ identity matrix and $x_{0 \mid 0} = E[x_0]$. The matrix Δ_k appearing in (1.3) is determined by

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