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ON THE MONOTONICITY OF THE OC OF AN SPRT[1]

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1. Introduction. This paper presents a theorem which may be of interest to anyone wishing to establish the monotonicity of the OC function of an SPRT. The same theorem may also be useful in finding bounds on the probability of acceptance when the actual value may be difficult or impossible to obtain.

In a well-known result, Lehmann [5], [6] established a sufficient condition for monotonicity when the observations form a sequence of independent random variables. This condition, simply stated, is that the distributions of the likelihood ratios be stochastically monotone. This is in turn satisfied if the family of densities possesses a monotone likelihood ratio. When the observations are not necessarily independent, Ghosh [3] has given a sufficient condition for monotonicity; namely, that the joint density of the observations possesses a monotone likelihood ratio. For further discussion in this area the reader is referred to the paper by Hall, Wijsman and Ghosh [4].

The theorem which is given in Section 2 of the paper is basically an extension of Lehmann's result to non-independent variables. It enables us to establish monotonicity for some problems in which Ghosh's condition is not met. In Section 3 two such examples of nonparametric type SPRT's are given.

2. Monotonicity of the OC. Let $\mathbf{X}_n = (X_1, \cdots, X_n)$ be a random vector with probability density function $p_{n\theta}(x_1, \cdots, x_n)$ which depends on the real parameter θ. Let

$$Z_i = \log \frac{p_{i\theta_1}(X_1, \cdots, X_i)}{p_{i\theta_0}(X_1, \cdots, X_i)} \qquad i = 1, 2, \cdots$$

be the sequence of log likelihood ratios which with the boundaries a, b define the SPRT of the hypotheses $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$.

We will use the following

LEMMA. *Let H be a Lebesgue measurable function defined on the real line which is nonincreasing and nonnegative. If F_1 and F_2 are two distribution functions on the real line such that $F_1(x) \geqq F_2(x)$ for all x then $\int H dF_1 \geqq \int H dF_2$.*

PROOF. The result follows easily by considering $\int H d(F_1 - F_2)$ and constructing a sequence of step functions increasing to H and finally by applying dominated convergence.

In the following theorem we consider two sequences of distribution functions $F = \{F_n\}$ and $G = \{G_n\}$ and compare the probability of accepting H_0 when the

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