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CENTERED VARIATIONS OF SAMPLE PATHS OF HOMOGENEOUS PROCESSES[1]

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1. Introduction. Let $X = \{X_t, t \geqq 0\}$ be a homogeneous stochastic process on the probability space $(\Omega, \mathscr{F}, P)$. In other words, the process X is assumed to have stationary independent increments given by the semigroup $\{\mu_t\}$. We wish to consider limits of the sums of the form

$$\sum_{[t_k \in \mathfrak{S}]} f(X_{t_{k+1}} - X_{t_k}) - b \tag{1.1}$$

where f is a certain function on the line and $\mathfrak{S} = \{t_0 < \cdots < t_n\}$ is a partition of $[0, t]$, over a sequence of partitions $\mathfrak{S}$ as the mesh of $\mathfrak{S}$ tends to zero.

The special case $f(x) = x^2$ is of special interest and has received much attention in the literature. It is easy to show, for example, that if X is a Brownian motion with no drift and $EX_t^2 = \sigma^2 t$, then the sum (1.1) converges in $L^2(\Omega, \mathscr{F}, P)$ to $\sigma^2 t$. If one assumes that the partitions are refining, a famous theorem of P. Lévy asserts that the convergence is almost sure.

Convergence in distribution of the sums (1.1) has been considered by Bochner [1] and Loève [3], though the latter paper studied limits where $X_{t_{k+1}} - X_{t_k}$ is replaced by a random variable X_{nk} of a triangular array of u.a.n. variables. In both the above papers, f was assumed to be at least continuous. Almost sure convergence of the sums (1.1) along a refining sequence of partitions was studied by Cogburn and Tucker [2], and they required f to be continuous and have a second derivative at 0, with $f(0) = 0$. In [4], we studied limits of (1.1) in the sense of convergence in probability and in $L^1(\Omega, \mathscr{F}, P)$ in the case where the centering term b vanishes. The function f was of rather general type, but the theorems held for a certain class of processes which included at least the non-Gaussian stable processes. In the present paper, we study limits for the same class of processes, but a different class of functions f, and the convergence in this case is in $L^2(\Omega, \mathscr{F}, P)$ or in probability.

2. Notation. Let $\{\mu_t\}$ be the weakly continuous convolution semigroup of probability measures on $(-\infty, \infty)$ associated with the process $\{X_t\}$. Let ν be the Lévy measure for $\{\mu_t\}$ so that

$$t^{-1}(x^2 \wedge 1)\mu_t(dx) \to \sigma^2 \delta_0(dx) + (x^2 \wedge 1)\nu(dx) \tag{2.1}$$

weakly as $t \to 0$, where σ^2 is the variance of the Gaussian component of X. (Note that in (2.1) of [4] it should be assumed that $f(x) = o(x^2)$ near 0, not $0(x^2)$ as stated.)

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