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ON THE ZEROS OF INFINITELY DIVISIBLE DENSITIES

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1. Introduction. Making use of a representation theorem for infinitely divisible (inf div) distributions on the nonnegative integers, which is implicit in [3], and its continuous analogue, which is implicit in [5], some properties are proved regarding the zeros of inf div probability density functions (pdf's) on $[0, \infty)$, both in the discrete and in the continuous case.

2. Representation theorems.

THEOREM 1. *A probability distribution* $\{p_n\}$ *on the nonnegative integers, with* $p_0 > 0$, *is* inf div *if and only if*

$$np_n = \sum_{j=0}^{n-1} p_j q_{n-j-1}, \tag{1}$$

where the q's satisfy

$$q_j \geqq 0 \ (j = 0, 1, 2, \cdots); \qquad \sum_{j=1}^{\infty} j^{-1} q_j < \infty. \tag{2}$$

PROOF. From Feller [1] (page 270 seq.) one easily obtains, that $\{p_n\}$ is inf div if and only if its generating function (pgf) $P(z)$ is of the form

$$P(z) = \exp\{-\lambda(1-R(z))\} \qquad (|z| \leqq 1),$$

where $\lambda > 0$ and $R(z)$ is the pgf of some distribution $\{r_n\}$ on the nonnegative integers. Equivalently we have, taking logarithmic derivatives,

$$P'(z) = P(z)Q(z) \qquad (|z| < 1),$$

where $Q(z) = \lambda R'(z)$.

Again equivalently,

$$np_n = \sum_{j=0}^{n-1} p_j q_{n-j},$$

where $q_n = \lambda(n+1)r_{n+1}$, with $\sum_1^{\infty}(n+1)^{-1}q_n = \lambda(1-r_0)$.

In the same way for general distributions on $[0, \infty)$ we have

THEOREM 2. *A distribution function* (df) $F(x)$ *on* $[0, \infty)$ *is* inf div *if and only if it satisfies*

$$\int_0^x u\,dF(u) = \int_0^x F(x-u)\,dP(u), \tag{3}$$

where P is non-decreasing, and

$$\int_1^{\infty} x^{-1}\,dP(x) < \infty. \tag{4}$$

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