The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 760–763

AN ITERATED LOGARITHM THEOREM FOR SOME WEIGHTED AVERAGES OF INDEPENDENT RANDOM VARIABLES[1]

BY R. JAMES TOMKINS

University of Saskatchewan, Regina

1. Introduction. In [1] Gaposhkin proved that, if $X_1, X_2, \cdots$ is a sequence of uniformly bounded, independent random variables (rv's), each with mean zero and variance one, then, for any $\alpha > 0$

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(1-k/n)^{\alpha} X_k}{(2n\log\log n)^{\frac{1}{2}}} = (2\alpha+1)^{-\frac{1}{2}} \text{ a.e.}$$

The purpose of this note is to present the following extension and generalization of Gaposhkin's result:

THEOREM. *Let $X_1, X_2, \cdots$ be a sequence of independent random variables. Suppose that there exists a sequence of positive numbers $c_n = o((\log\log n)^{-\frac{1}{2}})$ and a number $N > 0$ such that, if $n \geqq N$ then*

$$\exp\{(t^2/2n)(1-|t|c_n)\} \leqq E\exp\{tX_m/n^{\frac{1}{2}}\} \leqq \exp\{(t^2/2n)(1+|t|c_n/2)\} \tag{1}$$

for all $m \leqq n$, provided $|t|c_n \leqq 1$.

Let f be a real-valued function which is continuous on [0, 1], and define $S_n = \sum_{k=1}^{n} f(k/n)X_k$. Then

$$\limsup_{n\to\infty}(2n\log\log n)^{-\frac{1}{2}} S_n \geqq \|f\| \text{ a.e.} \tag{2}$$

where $\|f\| = (\int_0^1 f^2(x)\,dx)^{\frac{1}{2}}$.

Furthermore, any of the following conditions is sufficient to ensure that equality holds in (2):

(i) f is a polynomial:

(ii) f is an absolutely continuous *or* monotone function which can be written as a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on [0, 1] where $\limsup |a_n|^{1/n} = 1, \sum|a_n| < \infty, \sum a_n = f(1) = 0$.

(iii) f has a power series representation with radius of convergence greater than 1.

REMARK. Some results of a similar type have been obtained for independent, identically distributed random variables by Strassen [4].

Received January 26, 1970; revised April 21, 1970.

[1] The material contained in this paper is a revised version of the author's Ph.D. dissertation, which was prepared under the supervision of Y. S. Chow for Purdue University. This work was supported by the National Science Foundation under contract number NSF GP-07631.