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EQUICONVERGENCE OF MARTINGALES

BY EDWARD S. BOYLAN

Rutgers, The State University

Let $(X, \mathscr{F}, P)$ be a probability space, $\mathscr{F}_n$, $n = 1, 2, \cdots$, a sequence of subfields of $\mathscr{F}$ increasing (or decreasing) to a limit subfield $\mathscr{F}_\infty$ and $f \in L_1(X)$. It is a (by now) classical result of martingale theory that $\lim_{n\to\infty} E(f \mid \mathscr{F}_n) = E(f \mid \mathscr{F}_\infty)$ almost everywhere. (See [2] for details.) In all such convergence proofs, however, there is little investigation as to the rate of convergence to the limit. One would expect that knowledge regarding how quickly the subfields $\mathscr{F}_n$ "approach" the limit field $\mathscr{F}_\infty$ should yield some information regarding how well $E(f \mid \mathscr{F}_n)$ approximates $E(f \mid \mathscr{F}_\infty)$. Such information, depending only on the $\mathscr{F}_n$, is, in some sense, independent of the particular L_1 function f. In this paper we first define a pseudometric, D, on the set of subfields of $\mathscr{F}$ and then show that if $D(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ then rates of convergence of $E(f \mid \mathscr{F}_n)$ to $E(f \mid \mathscr{F}_\infty)$ can be given which are, in a sense to be defined below, independent of f.

DEFINITION 1. If $F \in \mathscr{F}$ and $\mathscr{F}'$ is a subfield of $\mathscr{F}$, let

$$d(F, \mathscr{F}') = \inf_{F' \in \mathscr{F}'} P(F \,\triangle\, F'),$$

where $F \triangle F' = (F - F') \cup (F' - F)$, the symmetric difference of F and F'.

DEFINITION 2. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two subfields of $\mathscr{F}$, let

$$d(\mathscr{F}_1, \mathscr{F}_2) = \sup_{F_1 \in \mathscr{F}_1} d(F_1, \mathscr{F}_2),$$

and $D(\mathscr{F}_1, \mathscr{F}_2) = d(\mathscr{F}_1, \mathscr{F}_2) + d(\mathscr{F}_2, \mathscr{F}_1)$.

THEOREM 1. *D is a pseudometric on the set of subfields of $\mathscr{F}$.*

PROOF. Clearly (a) $D(\mathscr{F}_1, \mathscr{F}_2) \geqq 0$; (b) $D(\mathscr{F}_1, \mathscr{F}_2) = D(\mathscr{F}_2, \mathscr{F}_1)$. It remains to show

(c) $D(\mathscr{F}_1, \mathscr{F}_3) \leqq D(\mathscr{F}_1, \mathscr{F}_2) + D(\mathscr{F}_2, \mathscr{F}_3)$. By symmetry, however, it suffices to show

(c′) $d(\mathscr{F}_1, \mathscr{F}_3) \leqq d(\mathscr{F}_1, \mathscr{F}_2) + d(\mathscr{F}_2, \mathscr{F}_3)$.

Suppose $d(\mathscr{F}_1, \mathscr{F}_2) = a$ and $d(\mathscr{F}_2, \mathscr{F}_3) = b$. To show that $d(\mathscr{F}_1, \mathscr{F}_3) \leqq a+b$ it suffices to show that for every $\varepsilon > 0$ and $F_1 \in \mathscr{F}_1$ there exists an $F_3 \in \mathscr{F}_3$ such that $P(F_1 \triangle F_3) \leqq a+b+\varepsilon$. Let $F_1 \in \mathscr{F}_1$. Since $d(\mathscr{F}_1, \mathscr{F}_2) = a$, there exists an $F_2 \in \mathscr{F}_2$ such that $P(F_1 \triangle F_2) \leqq a + \varepsilon/2$. Since $d(\mathscr{F}_2, \mathscr{F}_3) = b$, there is an $F_3 \in \mathscr{F}_3$ such that $P(F_2 \triangle F_3) \leqq b + \varepsilon/2$. The inclusion $F_1 \triangle F_3 \subset (F_1 \triangle F_2) \cup (F_2 \triangle F_3)$ implies

$$P(F_1 \triangle F_3) \leqq P(F_1 \triangle F_2) + P(F_2 \triangle F_3) \leqq a+b+\varepsilon.$$

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