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# ON BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS: I EXTINCTION PROBABILITIES[1]

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**0. Introduction.** Smith and Wilkinson (1969) (see also Smith (1968)) have formulated an intriguing model of a *branching process with random environment* (abbreviated as B.P.R.E.). The structure of the model can be delineated as follows. Let $\zeta_t$ be a discrete time ($t = 0, 1, 2, \cdots$) stochastic process of "environmental variables" taking values in some probability space $\Theta$. We suppose associated with each $\zeta \in \Theta$ is a probability generating function (p.g.f.)

$$\varphi_\zeta(s) = \sum_{j=0}^{\infty} p_j(\zeta)s^j, \qquad 0 \leqq s \leqq 1.$$

For each realization of the process $\zeta_t : \bar{\zeta} = (\zeta_0, \zeta_1, \zeta_2, \cdots)$ and the associated random sequence of p.g.f.'s, there evolves a population $Z_n$, $n = 0, 1, 2, \cdots$ governed by the laws of the standard temporally non-homogeneous branching process. Specifically, suppose $Z_0$ comprise the initial population number of the 0th generation. These individuals (alternatively called particles or objects) create progeny so that the population size at the first generation is

$$Z_1 = \sum_{i=1}^{Z_0} X_{1i}$$

where $X_{1i} (i = 1, 2, \cdots, Z_0)$ are independent random variables with p.g.f. $\varphi_{\zeta_0}(s)$. The second generation population number $Z_2$ is composed from the progeny of the $Z_1$ individuals each producing independently according to the p.g.f. $\varphi_{\zeta_1}(s)$. Proceeding in this way the $n+1$th generation population number $Z_{n+1}$ is determined as the cumulative progeny of the $Z_n$ particles of the $n$th generation each creating independently according to the p.g.f. $\varphi_{\zeta_n}(s)$. We shall call the process generated in this way $\{Z_n(\bar{\zeta}), n = 0, 1, 2, \cdots\}$ the branching process *conditioned* on the environment $\bar{\zeta}$. The population process $Z_n, n \geqq 0$ without specification of $\bar{\zeta}$ in advance is referred to as the branching process with random environment (B.P.R.E.).

Smith and Wilkinson limited themselves to the case where $\varphi_{\zeta_n}(s)$ is a sequence of independent and identically distributed random variables (i.i.d.). In that special case the process $Z_n, n \geqq 0$ is Markovian. These authors devoted their efforts exclusively to ascertaining conditions for certain or noncertain extinction of the B.P.R.E. model.

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