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# IRREDUCIBLE MARKOV SHIFTS

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This paper classifies irreducible finite state Markov shifts up to isomorphism, showing that such a shift is isomorphic to a direct product of a rotation and a Bernoulli shift. This extends the result of Friedman Ornstein [3] that a mixing Markov shift is isomorphic to a Bernoulli shift.

The definition and various properties of Markov shifts are given in Billingsley [1] and will be summarized here.

Suppose $S$ is a finite set, say $S = \{1, \cdots, s\}$, and $P = (P_{ij})$, $i, j \in S$, is a stochastic transition matrix, that is, $P$ is a nonnegative matrix each of whose rows sums to one. A path (of length $n$) from $i$ to $j$ in $S$ is a sequence $i_0, i_1, \cdots, i_n$, such that $i_0 = i$, $i_n = j$, and

$$P_{i_0 i_1} P_{i_1 i_2} \cdots, P_{i_{n-1} i_n} > 0 .$$

It will be assumed throughout this paper that $P$ is *irreducible*, that is, given any $i, j$ is $S$, there is a path from $i$ to $j$.

A Markov shift $\tau$ is constructed from $P$ as follows: Let $X$ be the set of all functions from the integers $Z$ into $S$, and $\mathfrak{B}_0$ the $\sigma$-algebra generated by the cylinder sets

$$C = C(i_k, \cdots, i_n) = \{x \mid x_i = i_j, k \leqq j \leqq n\} \tag{1}$$

for $k, n \in Z$, $k \leqq n$. The matrix $P$, being irreducible, determines a unique probability vector $\pi$ such that $\pi P = \pi$. There is a unique measure $m$ on $\mathfrak{B}$ such that if $C$ is given by (1), then

$$m(C) = \pi_k \prod_{j=k}^{n-1} P_{i_j i_{j+1}} .$$

The shift $\tau$, defined by $(\tau x)_n = x_{n+1}$, $n \in Z$, is an invertible, ergodic, measure-preserving transformation on $(X, \mathfrak{B}, m)$, where $\mathfrak{B}$ is the $m$-completion of $\mathfrak{B}_0$. This transformation $\tau$ is called the *Markov shift* with transition matrix $P$ and stationary vector $\pi$. In the case where the rows of $P$ are identical (and hence equal to $\pi$), the shift $\tau$ is called a *Bernoulli shift* with distribution $\pi$.

The *period* $\nu = \nu(P)$ of $P$ is the greatest common divisor of the lengths of cycles, that is, paths from $i$ to $i$, $i \in S$. If $\nu(P) = 1$, then $\tau$ is mixing, while if $\nu > 1$, then $\tau^\nu$ is not ergodic. N.A. Friedman and D.S. Ornstein [3] have proved the following theorem:

THEOREM 1. *A mixing Markov shift is isomorphic to a Bernoulli shift.*

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