# PROBLEMS IN PROBABILITY THEORY

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**1. Introduction.** The following survey of problems in probability theory has been written for the occasion of the Princeton Bicentennial Conference on "The Problems of Mathematics," Dec. 17–19, 1946. It is strictly confined to the purely mathematical aspects of the subject. Thus all questions concerned with the philosophical foundations of mathematical probability, or with its ever increasing fields of application, will be entirely left out.

No attempt to completeness has been made, and the choice of the problems considered is, of course, highly subjective. It is also necessary to point out explicitly that the literature of the war years has only recently—and still far from completely—been available in Sweden. Owing to this fact, it is almost unavoidable that this paper will be found incomplete in many respects.

## I. FUNDAMENTAL NOTIONS

**2. Probability distributions.** From a purely mathematical point of view, probability theory may be regarded as the theory of certain classes of *additive set functions*, defined on spaces of more or less general types. The basic structure of the theory has been set out in a clear and concise way in the well-known treatise by Kolmogoroff [53]. We shall begin by recalling some of the main definitions. Note that the word *additive*, when used in connection with sets or set functions, will always refer to a *finite or enumerable* sequence of sets.

Let $\omega$ denote a variable point in an entirely arbitrary space $\Omega$, and consider an additive class $C$ of sets in $\Omega$, such that the whole space $\Omega$ itself is a member of $C$. Further, let $P(S)$ be an additive set function, defined for all sets $S$ belonging to the class $C$, and suppose that

$$P(S) \geqq 0 \text{ for all } S \text{ in } C,$$
$$P(\Omega) = 1.$$

We shall then say that $P(S)$ is a *probability measure*, which defines a *probability distribution in* $\Omega$. For any set $S$ in $C$, the quantity $P(S)$ is called the *probability* of the *event* expressed by the relation $\omega \subset S$, i.e. the event that the variable point $\omega$ takes a value belonging to $S$. Accordingly we write

$$P(S) = P(\omega \subset S).$$

Suppose now that $\omega' = g(\omega)$ is a function of the variable point $\omega$, defined throughout the space $\Omega$, the values $\omega'$ being points of another arbitrary space $\Omega'$. Let $S'$ be a set in $\Omega'$ and denote by $S$ the set of all points $\omega$ such that $\omega' = g(\omega)$ belongs to $S'$. Whenever $S$ belongs to $C$, we define a set function $P'(S')$ by writing

$$P'(S') = P(S).$$