ON LIMIT THEOREMS FOR GAUSSIAN PROCESSES[1]

By John Lamperti

Dartmouth College

1. Introduction. This paper has two purposes. It consists of a new attack, using different methods, on some problems concerning large excursions of Gaussian processes which were studied by M. Kac and D. Slepian in [1], and it may be considered a technical contribution to this topic since some new and perhaps simpler proofs are given under less restrictive assumptions than theirs. These problems (among others) have also been studied by Volkonski and Rozanov in [4], but again under more stringent assumptions than we shall require. On the other hand, this paper is intended to provide a further illustration of the point made in [2] that choice of the proper topology may be of great importance in connection with convergence of stochastic processes. It seems likely that the method used below will have other applications in the study of Gaussian processes, and it is this hope which is responsible for the rather general title I have chosen.

2. The limiting process. Let $\{x(t)\}$ denote a real, continuous, stationary, Gaussian stochastic process with mean 0 and covariance function satisfying

$$\rho(t) = E(x(s)x(t+s)) = 1 - \tfrac{1}{2}\alpha t^2 + o(t^2) \tag{1}$$

for small t. Kac and Slepian studied the behavior of $x(t)$, conditioned so that $x(0) = a$, as a tends to $+\infty$; the particular question of greatest interest is the distribution of the time until the next return to the level a in the cases when $x'(0) > 0$. They also discussed several different interpretations of the conditional probabilities, which led to somewhat different results. In this paper, however, we shall work at first with the ordinary notion of conditional probabilities and densities (equivalent to those called "vertical window" in [1]); this considerably simplifies the problem since the conditioned processes are then still Gaussian. Later on it will be shown how the results can all be carried over to the other types of conditioning considered by Kac and Slepian.

Let us now define

$$\Delta(t, \theta) = (x(\theta t) - a)/\theta, \tag{2}$$

where

$$\theta = (2\pi/\alpha)^{\frac{1}{2}} a^{-1} \quad \text{for} \quad a > 0; \tag{3}$$

obviously $\Delta(0, \theta) = 0$ and $\theta \to 0$ are respectively equivalent to $x(0) = a$ and

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